ՀԱՅԱՍՏԱՆԻ ԳԻՏՈՒԹՅՈՒՆՆԵՐԻ ԱԶԳԱՅԻՆ ԱԿԱԴԵՄԻԱՅԻ ՏԵՂԵԿԱԳԻՐ
ИЗВЕСТИЯ НАЦИОНАЛЬНОЙ АКАДЕМИИ НАУК АРМЕНИИ

Մեխանիկա 78, №3-4, 2025 Механика

УДК 531.8 DOI: 10.54503/0002-3051-2025.78.3-4-90

ENERGY-OPTIMAL COMBINED CONTROL OF AN ELECTROMECHANICAL MANIPULATOR UNDER CONTROL VOLTAGE CONSTRAINT

Avetisyan V.V.

Keywords: electromechanical manipulator, optimal control, energy consumption

Аветисян В.В.

Оптимальное по энергозатратам комбинированное управление электромеханическим манипулятором с ограничением на управляющее напряжение

Ключевые слова: электромеханический манипулятор, оптимальное управление, энергозатраты

Рассматривается задача построения оптимального по энергозатратам управления однозвенным электромеханическим манипулятором с ограничением на управляющее напряжение. Проанализированы различные структуры оптимального управления, вытекающие из принципа максимума Понтрягина. Показано, что при больших временах перехода оптимальным является управление, полученное без учёта ограничений, тогда как при малых временах возможны комбинированные управления двух различных структур. Однако лишь одно из них, в определённом диапазоне изменения конечного времени, является единственно допустимым и, следовательно, оптимальным. Разработан алгоритм построения такого комбинированного управления, реализуемость которого подтверждается результатами численного моделирования.

Ավետիսյան Վ.Վ.

Էլեկտրամեխանիկական մանիպուլյատորի ըստ էներգածախսերի օպտիմալ համակցված ղեկավարումը՝ ղեկավարող լարման սահմանափակմամբ

Հիմնաբառեր: էլեկտրամեխանիկական մանիպուլյատոր, օպտիմալ ղեկավարում, էներգածախսեր

Դիտարկվում է միաթև էլեկտրամեխանիկական մանիպուլյատորի օպտիմալ ըստ էներգածախսերի ղեկավարման խնդիրը՝ ղեկավարող լարման սահմանափակման առկայության պայմաններում։ Վերլուծվել են Պոնտրյագինի մաքսիմումի սկզբունքից բխող օպտիմալ ղեկավարման տարբեր կառուցվածքները։ Ցույց է տրվել, որ տեղափոխման բավականին մեծ ժամանակահատվածի դեպքում օպտիմալ է այն ղեկավարումը, որը գտնվել է առանց սահմանափակումները հաշվի առնելու, մինչդեռ փոքր ժամանակահատվածի դեպքում հնարավոր են երկու տարբեր կառուցվածքով համակցված ղեկավարումներ։ Սակայն դրանցից միայն մեկն է հանդիսանում միակ թույլատրելին և, հետևաբար, օպտիմալը՝ վերջնական ժամանակի փոփոխման որոշակի տիրույթում։ Մշակվել է համապատասխան համակցված ղեկավարման կառուցման ալգորիթմը, որի իրագործելիությունը հաստատվել է թվային մոդելավորմամբ։

The problem of designing energy-optimal control for a single-link electromechanical manipulator with a constraint on the control voltage is considered. Various structures of optimal control, derived from Pontryagin's maximum principle, are analyzed. It is shown that for large transition times, the optimal control is the one obtained without accounting for constraints, whereas for small transition times, combined controls of two different structures are possible. However, only one of them, within a certain range of terminal time variation, is uniquely admissible and, consequently, optimal. An algorithm for constructing such combined control is developed, and its feasibility is confirmed by numerical simulation results.

Introduction. Modern electromechanical manipulators are widely used in automated control systems, where the issues of energy efficiency, accuracy, and compliance with technological constraints play a crucial role [1–4]. When moving a manipulator under specified boundary conditions, the control problem for the electric drive arises in such a way as to minimize energy (in particular, thermal) losses while satisfying constraints on the control input [5]. These problems are especially relevant when operating with limited power sources or when it is necessary to extend the service life of the electric motor [6]. Similar problems have been repeatedly addressed in the scientific literature. For instance, [7] proposes a method for constructing energy – optimal trajectories for robot manipulators with holonomic constraints, based on minimizing energy consumption while considering control force limitations. In [8], the problem of optimal control of a flexible single – link manipulator is studied, taking into account inertial properties and vibrations; special attention is given to the energy functional and to control structures that include both bang – bang and smooth control segments. Moreover, in the context of modern high degree of freedom robots, [9] develops an energy – efficient trajectory planning strategy aimed at reducing energy consumption and improving motion smoothness. Various problems of minimizing energy consumption during transport and search operations performed by electromechanical manipulators are addressed in [10–14] and [15,16], respectively.

In this paper, we address the problem of designing energy – optimal control for a single-link electromechanical manipulator subject to a constraint on the control voltage. The focus is on identifying the structure of the optimal control for different transition durations and on developing an algorithm for its construction based on Pontryagin's maximum principle.

1. Mathematical model of the electromechanical manipulator and problem statement. Let us consider a simple model of an electromechanical manipulator consisting of a separately excited direct current motor, a gearbox, and an arm with a load, rotating in the horizontal plane. Such a system can be regarded as a model of the simplest manipulator with a single degree of freedom. The motion of the described electromechanical system, in the given form, is defined by the following differential equation [17]:

$$R\left(I + Jn^2\right)\ddot{\varphi} + k^2 n^2 \dot{\varphi} = knu\,. \tag{1.1}$$

Here φ is the angular position of the manipulator arm relative to a fixed axis; I is the moment of inertia of the arm (including the driven gear of the gearbox) with respect to the axis of rotation; J is the moment of inertia of the motor armature (including the driving gear) relative to its rotation axis; n is the gear ratio of the gearbox; R is the electrical resistance of the armature winding; k is a motor constant; u is the control voltage applied to the motor input.

Equation (1.1) is valid under the assumption that the electromagnetic time constant of the system is significantly smaller than both the duration of the robot's working operation and the time required for the electric motor to reach a steady – state rotational mode under constant voltage. For many industrial electromechanical robots, this assumption is generally satisfied [1].

Let us now consider the problem of optimal control for system (1.1).

It is required to find a programmed law of variation of the control voltage that transfers the manipulator, described by equation (1.1), from a given initial rest state

$$\varphi(0) = \varphi^0, \qquad \dot{\varphi}(0) = 0 \tag{1.2}$$

to a specified terminal rest state (1.3) at a fixed moment of time T

$$\varphi(T) = \varphi^T, \qquad \dot{\varphi}(0) = 0 \tag{1.3}$$

under the condition that the control satisfies the constraint

$$|u(t)| \le U \tag{1.4}$$

where $U > 0$ is the prescribed maximum allowable value of the control voltage and minimizes the functional

$$Q = R^{-1}\int_0^T \left(u - k\dot{\varphi}\right)^2 dt\,, \tag{1.5}$$

The functional (1.5) characterizes the total heat dissipation in the rotor winding of the electric motor during the control process [18]. The terminal time T is fixed, but its value is refined in the course of solving the problem.

In equations (1.1)–(1.5), we transition to dimensionless units (with primes subsequently omitted) and introduce the following notation and new variables:

$$\begin{gathered} t' = t/\overline{T}, \quad u' = u/U, \quad k' = kn/\left(U\overline{T}\right), \quad R' = RA/\left(knU\overline{T}^2\right), \\ Q' = \overline{T}^2 A^{-1} Q, \quad Q_1' = Q'k'R', \quad A = I + Jn^2, \quad \varphi_1 = \varphi - \varphi_1^0, \quad \varphi_2 = \dot{\varphi}, \end{gathered} \tag{1.6}$$

where $\overline{T}$ is the characteristic time of the robot's working operation, taken as the unit of measurement and to be specified below.

Then, relations (1.1)–(1.5) take the following form:

$$\dot{\varphi}_1 = \varphi_2\,, \ \ \dot{\varphi}_2 = uR^{-1} - kR^{-1}\varphi_2\,, \tag{1.7}$$

$$\varphi_1(0) = 0\,, \ \ \varphi_2(0) = 0\,, \tag{1.8}$$

$$\varphi_1(T) = \varphi_1^T\,, \ \ \varphi_2(T) = 0\,, \tag{1.9}$$

$$|u(t)| \le 1\,, \tag{1.10}$$

$$Q_1 = \int_0^T \left(u - k\dot{\varphi}\right)^2 dt\,. \tag{1.11}$$

Thus, the problem formulated above is reduced to the equivalent problem (1.7)–(1.11). Without loss of generality, we assume that in (1.9) $\varphi_1^T > 0$.

2. Analysis of the admissibility and optimality of various control structures. To solve problem (1.7)–(1.11), we apply the Pontryagin's maximum principle [19]. The Hamiltonian of the system (1.7)–(1.11) has the form

$$H = -(u - k\varphi_2)^2 + p_1\varphi_2 + p_2(uR^{-1} - kR^{-1}\varphi_2)\,, \tag{2.1}$$

where p_1, p_2 are the adjoint variables determined from the equations

$$\dot{p}_1 = -H_{\varphi_1} = 0, \quad \dot{p}_2 = -H_{\varphi_2} = -2ku + 2k^2\varphi_2 - p_1 + kR^{-1}p_2, \quad 0 \le t \le T. \tag{2.2}$$

The maximum of the function (2.1), without taking into account the constraint (1.10), is found from the condition that the derivative of H with respect to u is zero:

$$H_u(t) = -2(u(t) - k\varphi_2(t)) + p_2(t)R^{-1} = 0, \quad 0 \le t \le T \tag{2.3}$$

and is attained at the point

$$u^*(t) = k\varphi_2(t) + p_2(t)(2R)^{-1}. \tag{2.4}$$

The variables φ_2 and p_2 in (2.4) are obtained by solving the boundary value problem (1.7)–(1.9), (2.4) with the optimal control (2.4) and have the form

$$\varphi_1(t) = \left(-2T^{-3}t^3 + 3T^{-2}t^2\right)\varphi_1^T, \qquad \varphi_2(t) = \left(-6T^{-3}t^2 + 6T^{-2}t\right)\varphi_1^T, \tag{2.5}$$

$$p_1(t) = 24R^2T^{-2}\varphi_1^T, \quad p_2(t) = -24T^{-3}R^2\varphi_1^T t + 12T^{-2}R^2\varphi_1^T. \tag{2.6}$$

According to (2.4)–(2.6), the optimal control (2.4) takes the form

$$u^*(t) = -6kT^{-3}\varphi_1^T t^2 + 6T^{-2}(k - 2RT^{-1})\varphi_1^T t + 6RT^{-2}\varphi_1^T. \tag{2.7}$$

From (2.7) it follows that $u^*(t)$ is a concave quadratic function that reaches its maximum at the point

$$t_* = T/2 - k^{-1}R. \tag{2.8}$$

The corresponding maximum value is

$$u^*(t_*) = 1.5kT^{-1}\varphi_1^T + 6R^2k^{-1}T^{-3}\varphi_1^T. \tag{2.9}$$

Note that at the ends of the interval $[0,T]$, the function $u^*(t)$ takes the values

$$u^*(0) = 6T^{-2}R\varphi_1^T, \quad u^*(T) = -6T^{-2}R\varphi_1^T, \tag{2.10}$$

which is important when verifying the fulfillment of the constraint $|u(t)| \le 1$.

As follows from (2.8) and (2.9), if $t_* \le 0$, then the function $u^*(t)$ is monotonically decreasing over the entire interval $[0,T]$. If $0 < t_* < T$, then $u^*(t)$ increases monotonically on the interval $[0,t_*]$ and decreases monotonically on $[t_*,T]$. In both cases the maximum value $u^*(t_*)$, regarded as a function of T, decreases monotonically on the interval $[0,+\infty)$, taking values from $+\infty$ to 0. Therefore, there exists a value $T = T' \in (0,+\infty)$ such that the maximum value (2.9) reaches the admissible boundary of control: $u^*(t_*) = 1$. From this equality, we find [20]

$$T' = \sqrt[3]{-(q/2) + \sqrt{D}} + \sqrt[3]{-(q/2) - \sqrt{D}} - b/(3a), \tag{2.11}$$

$$p = -b^2 / \left(3a^2\right) = -3k^2(\varphi_1^T)^2 / 4,$$
$$q = 2b^3 / \left(27a^3\right) + d / a = -k^3(\varphi_1^T)^3 / 4 - 6R^2\varphi_1^T / k,$$
$$b / \left(3a\right) = -k\varphi_1^T / 2,$$
$$D = \left(p/3\right)^3 + \left(q/2\right)^2 = 3k^2R^2(\varphi_1^T)^4 / 4 + 9R^4(\varphi_1^T)^2 / k^2 > 0.$$

Thus, for a fixed value of φ_1^T and for any T satisfying the inequality $T' \le T < \infty$, the control defined by formula (2.7) transfers system (1.7) from the initial state of rest (1.8) to the final state of rest (1.9), satisfies the constraint (1.10) and minimizes the functional (1.11).

Taking into account the constraint (1.10) and according to the maximum principle, the optimal control that maximizes the Hamiltonian (2.3) is determined as follows:

$$u(t) = \begin{cases} 1, & \text{если } u^*(t) > 1, \quad (\text{т.е. } H_u(t) > 0), \\ u^*(t), & \text{если } \left|u^*(t)\right| \le 1, \ (\text{т.е. } H_u(t) = 0), \\ -1, & \text{если } u^*(t) < -1, \ (\text{т.е. } H_u(t) < 0), \end{cases} \qquad 0 \le t \le T', \tag{2.12}$$

where $u^*(t)$ denotes the optimal control (2.7) derived from the maximum condition (2.3) without considering the constraints.

Let us examine the possible control structures (2.11) that may qualify as optimal.

From the concavity property of the function $u^*(t)$, it follows that for $0 < T < T'$, the maximum value of the control exceeds the constraint: $u^*(t) > 1$. Let us denote by

$$T^* = \sqrt{6R\varphi_1^T} \tag{2.13}$$

– the critical value of time at which the function $u^*(t)$ first reaches the constraint at the endpoints of the interval $[0, T]$.

Then, the following cases are possible:

if $T^* < T < T'$, then $u^*(t_*) > 1$ and $u^*(0) = -u^*(T) = 6T^{-2}R\varphi_1^T < 1$, (2.14)

if $0 < T \le T^*$, then $u^*(t_*) > 1$ and $u^*(0) = -u^*(T) = 6T^{-2}R\varphi_1^T \ge 1$. (2.15)

Definition. A control $u(t)$ is said to be admissible for problem (1.7)–(1.11) if it belongs to the class of piecewise – continuous functions $KC[0, T]$, satisfies the constraint (1.10), and the corresponding solution of the system (1.7), (1.8) satisfies the terminal conditions (1.9).

In accordance with the definition of admissible control and taking into account cases (2.14) and (2.15), let us represent control (2.12) in more specific structures corresponding to these cases:

$$u(t)=\begin{cases} u^*(t), & t\in[0,t_1] \text{ and } H_u(t)=0,\\ 1, & t\in(t_1,t_2) \text{ and } H_u(t)>0,\\ u_1(t), & t\in[t_2,T] \text{ and } H_u(t)=0.\end{cases} \qquad T^*\le T\le T', \tag{2.16}$$

$$u(t)=\begin{cases} 1, & t\in[0,t_1) \text{ and } H_u(t)>0,\\ u_2(t), & t\in[t_1,t_2] \text{ and } H_u(t)=0,\\ -1, & t\in(t_2,T] \text{ and } H_u(t)<0,\end{cases} \qquad 0\le T\le T^*. \tag{2.17}$$

In (2.16) and (2.17), the moments t_1 and t_2 denote the switching points between control regimes that arise when the optimal control reaches its boundary values: $u^*(t)=\pm 1$. The functions $u_1(t)$ and $u_2(t)$ are defined on the intervals $[t_2,T]$ and $[t_1,t_2]$, respectively, and satisfy the conditions $|u_{1,2}(t)|\le 1$ and $H_u(t)=0$. These functions may partially coincide with $u^*(t)$, but in the general case they are chosen to ensure admissibility.

We use the results of work [20], in which, for given values of $R>0$, $k>0$, $\varphi_1^T>0$, the conditions for admissibility of controls (2.16) and (2.17) were established.

Let us denote by

$$T_{\min}=\frac{R}{k^2}\ln\left\{\left[1+\sqrt{1-\exp\left(-\frac{k\varphi_1^T}{R}\right)}\right]\left[1-\sqrt{1-\exp\left(-\frac{k\varphi_1^T}{R}\right)}\right]^{-1}\right\} \tag{2.18}$$

– the minimum transition time of system (1.7) from state (1.8) to state (1.9) in the time-optimal control problem with constraint (1.10) [21].

According to the results of [20] and taking into account cases (2.14) and (2.15), the admissibility of controls (2.16) and (2.17) is determined by the relationships between the quantities T' (2.11), T^* (2.13) and $T_{\min}$ (2.18). Two cases are possible.

Case A. If $T_{\min}<T^*<T'$, then the structure (2.17) is inadmissible for $T\in(T_{\min},T^*]$, while the structure (2.16) is admissible for $T\in(T^*,T')$.

Case B. If $T^*\le T_{\min}<T'$, then for all $T\in(T_{\min},T')$ the structure (2.17) is inadmissible, and the structure (2.16) is admissible. In the latter case, the optimality of structure (2.16) in problem (1.7)–(1.11) is also proven.

In summary:

– if $T'<T<\infty$, then control (2.7) is optimal in problem (1.7)–(1.11);

– if $T\in(T_{\min},T')$, then the optimal control is the combined control (2.16), for which an algorithm is presented in Section 3;

– if $T\in(0,T_{\min}]$, then no admissible controls exist.

3. Algorithm for constructing the optimal combined control and numerical simulation. Let us proceed to the construction of the combined control (2.16) and the corresponding dependencies of $\varphi_1(t)$, $\varphi_2(t)$ for problem (1.7)–(1.11), relying on the conditions of admissibility and optimality established above. Let the problem parameters be fixed: terminal time T, coefficients R, k and the terminal coordinate value $\varphi_1(T)=\varphi_1^T$.

We integrate system (1.7) with the initial condition (1.8) over the interval $[0,t_1]$, where the control is given by the function $u(t)=u^*(t)$ obtained without considering the control constraint. At the moment

$$t_1 = T/2 - R/k - \sqrt{T^2/4 - T^3/(6k\varphi_1^T) + R^2/k^2}\ , \tag{3.1}$$

when the control (2.7) reaches the boundary value $u^*(t)=1$, we obtain the values of the phase variables

$$\begin{aligned}\varphi_1(t_1) &= \left(-2T^{-3}t_1^3 + 3T^{-2}t_1^2\right)\varphi_1^T = \varphi_1^{(1)},\\ \varphi_2(t_1) &= \left(-6T^{-3}t_1^2 + 6T^{-2}t_1\right)\varphi_1^T = \varphi_2^{(1)}.\end{aligned} \tag{3.2}$$

On the next interval (t_1,t_2), by integrating system (1.7) with initial conditions (3.2) under constant control $u(t)=1$, we obtain

$$\begin{aligned}\varphi_1(t) &= k^{-1}(t-t_1) + Rk^{-1}(k^{-1}-\varphi_2^{(1)})[\exp(-kR^{-1}(t-t_1))-1] + \varphi_1^{(1)},\\ \varphi_2(t) &= k^{-1} - (k^{-1}-\varphi_2^{(1)})\exp(-kR^{-1}(t-t_1)).\end{aligned} \tag{3.3}$$

The values of the phase variables at the point t_2 will be denoted, respectively, by

$$\varphi_1(t_2)=\varphi_1^{(2)},\quad \varphi_2(t_2)=\varphi_2^{(2)}\ . \tag{3.4}$$

On the final interval $[t_2,T]$ the control has the form (2.4)

$$u_1(t) = k\varphi_2(t) + p_2(t)/(2R) \quad p_2(t) = -A(t-t_2)+B\,, \tag{3.5}$$

where $p_2(t)$ is the solution of the second adjoint equation (2.2), A and B are integration constants to be determined.

Substitute expression (3.5) into system (1.7)

$$\dot\varphi_1 = \varphi_2,\quad \dot\varphi_2 = \frac{-A(t-t_2)+B}{2R^2}\ . \tag{3.6}$$

Integrating (3.6) with the initial conditions (3.4), we obtain

$$\varphi_1(t) = -\frac{A}{12R^2}(t-t_2)^3 + \frac{B}{4R^2}(t-t_2)^2 + \varphi_2^{(2)}(t-t_2) + \varphi_1^{(2)}, \tag{3.7}$$

$$\varphi_2(t) = -\frac{A}{4R^2}(t-t_2)^2 + \frac{B}{2R^2}(t-t_2) + \varphi_2^{(2)}\ . \tag{3.8}$$

Taking into account (3.8), the control (3.5) can be written as

$$u_1(t) = -\frac{kA}{4R^2}(t-t_2)^2 + \frac{kB-RA}{2R^2}(t-t_2) + \frac{B}{2R} + k\varphi_2^{(2)} \tag{3.9}$$

For the admissible control $u_1(t)$ on $\left[t_2, T\right]$ the following conditions must be satisfied:
1) boundary conditions (1.9);
2) continuity of the control at the point t_2: $u_1(t_2) = 1$;
3) concavity and monotonicity: $A > 0$, $B < RA/k$;
4) compliance with the constraint at the end: $u_1(T) \geq -1$.

From 2) we obtain $u_1(t_2) = k\varphi_2^{(2)} + \dfrac{B}{2R} = 1$ or equivalently:

$$B = 2R(1 - k\varphi_2^{(2)}) . \tag{3.10}$$

After substituting (3.10) into formulas (3.7) and (3.8) for $t = T$, from condition $\varphi_2(T) = 0$ (1.9) we find an explicit expression for

$$A = \frac{4R(1-k\varphi_2^{(2)})}{(T-t_2)} + \frac{4R^2\varphi_2^{(2)}}{(T-t_2)^2}, \tag{3.11}$$

which is then substituted into condition $\varphi_1(T) = \varphi_1^T$ (1.9).

As a result, we obtain an equation with respect to a single parameter t_2:

$$\frac{(1-k\varphi_2^{(2)})}{6R}(T-t_2)^2 + \frac{2\varphi_2^{(2)}}{3}(T-t_2) + \varphi_1^{(2)} - \varphi_1^T = 0 . \tag{3.12}$$

It has two real roots

$$t_2 = T - \frac{-2R\varphi_2^{(2)} \pm \sqrt{(2R\varphi_2^{(2)})^2 - 6R(1-k\varphi_2(t_2))(\varphi_1^{(2)} - \varphi_1^T)}}{1-k\varphi_2(t_2)}, \tag{3.13}$$

but only one – with the positive sign in front of the radical – satisfies conditions 1) – 4). This choice will also be confirmed in the numerical example below.

When these conditions are satisfied, the problem is considered solved, and the obtained control is deemed admissible and optimal within the chosen structure.

Let's present a numerical example of the implementation of the proposed algorithm for constructing the optimal control (2.16). Suppose the electromechanical manipulator is characterized by the following dimensional parameters [1], which enter into equation (1.1):

$$I = 5.9\,\mathrm{kg\cdot m^2},\quad J = 2.45\cdot 10^{-4}\,\mathrm{kg\cdot m^2},\quad n = 163, \tag{3.14}$$
$$R = 3.6\,\mathrm{Om},\quad k = 0.233\,\mathrm{N\cdot m/A},\quad U = 110\,\mathrm{V}.$$

When switching to nondimensional variables (1.6), we choose as the unit of time the quantity $\overline{T} = nkU^{-1} \approx 0.345s$, equal to the time it takes the manipulator arm to rotate by one radian when moving at a steady angular velocity $\dot{\varphi} = (nk)^{-1} U \approx 2.9 s^{-1}$ [1]. The nondimensional parameters of equation (1.7) then take the values:

$$R \approx 0.09, \quad k \approx 1. \tag{3.15}$$

With an initial angle of zero $\varphi_1(0) = 0$ rad, we set the final rotation angle corresponding to condition (1.9) equal to

$$\varphi_1^T = 1 \text{ rad.} \tag{3.16}$$

Construction of the optimal control (2.16) and the corresponding phase trajectories $\varphi_1(t),\ \varphi_2(t)$ is carried out in the following sequence:

1. Using the given parameters (3.15), (3.16) and formulas (2.11), (2.13), and (2.18), determine the values: $T^* = 0.735$, $T' = 1.52$, $T_{\min} = 1.125$. Since $T^* < T_{\min} < T'$, the case B considered in Section 2 is realized, in which $T \in (T_{\min}, T') = (1.125, 1.52)$.

2. Choose $T = 1.5 \in (1.125, 1.52)$.

3. For the selected value of T, construct the optimal control profile (2.16). In this process, the switching time $t_1 = 0.57$ is calculated using formula (3.1), and to determine the switching time t_2 the following procedure is applied.

First, using formulas (3.10) and (3.11), the coefficients $A = 0.0594$, $B = 0.0016$ are calculated so that all the admissibility conditions for the control $u_1(t)$ (3.9), formulated in items 1)–4) of Section 3, are satisfied. Then, the switching time t_2 is determined as the root of equation (3.13).

Figure 1 shows the graph of the function $F(t_2)$, defined by equation (3.13) for the numerical values of the parameters (3.15), (3.16), $T = 1.5$. The function $F(t_2)$ has the form:

$$F(t_2) = 1.5 - t_2 - \frac{-0.18\varphi_2^{(2)} + \sqrt{0.0324(\varphi_2^{(2)})^2 - 0.54(1 - \varphi_2^{(2)})(\varphi_1^{(2)} - 1)}}{1 - \varphi_2^{(2)}}$$

where the values $\varphi_1^{(2)} = \varphi_1^{(2)}(t_2)$, $\varphi_2^{(2)} = \varphi_2^{(2)}(t_2)$ are determined by formulas (3.3) and (3.4) at $t = t_2$.

From the graph in Figure 1, it is seen that the equation $F(t_2) = 0$ on the interval $(0.57, 1.5)$ has a unique solution. The result of the numerical solution is $t_2^* = 0.7373$, which confirms the correctness of the chosen parameters A, B and the uniqueness of the constructed optimal control.

Figures 2–4 show the graphs of the optimal control (2.16) and the dependencies $\varphi_1(t)$, $\varphi_2(t)$, obtained at $T = 1.5$. The control $u(t)$ (fig. 2) has a combined structure: on the initial interval $[0, t_1]$ and the final interval $[t_2, T]$ it varies according to a smooth (parabolic) law, while on the middle interval $[t_1, t_2]$ it takes a constant value $u(t) = 1$,

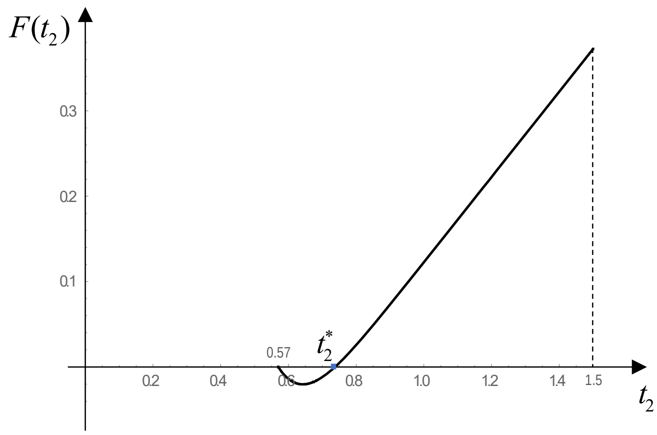

Fig. 1

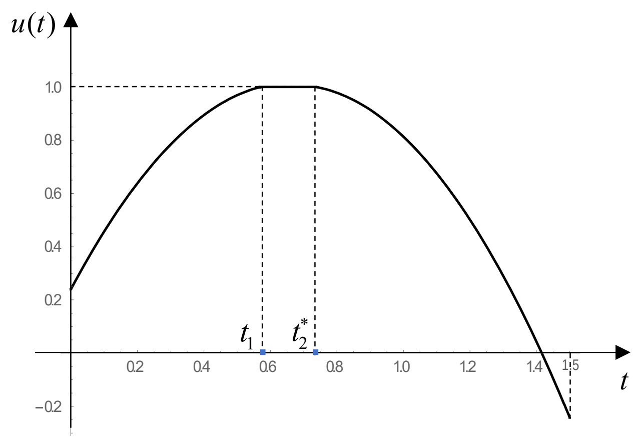

Fig. 2

satisfying the constraint $|u(t)| \le 1$ for all $t \in [0, T]$.

Based on the results of numerical simulation, the following values were obtained:

$\varphi_1(1.5) = 1.00024$ rad, $\varphi_2(1.5) = 0.00099$ rad,

which correspond to an accuracy on the order of 10^{-3}, upon reaching the final state.

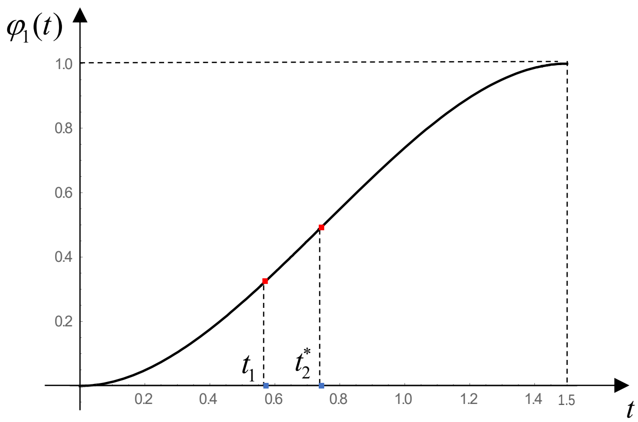

Fig. 3

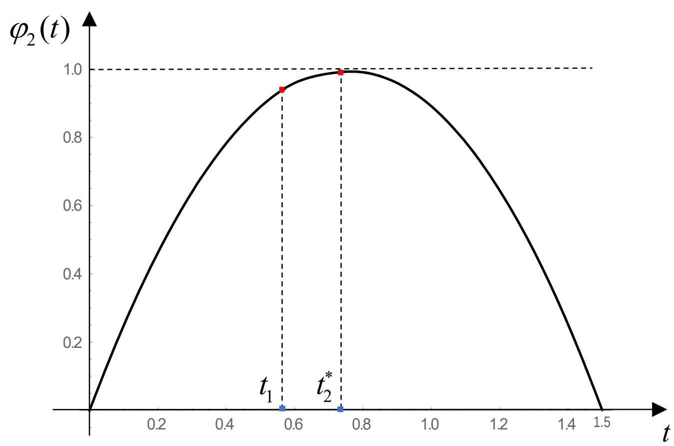

Fig. 4

Since the moment is uniquely determined from equation (3.15), the corresponding admissible control (2.16), satisfying constraint (1.10) and transferring system (1.7) from the rest state (1.8) to the final rest state (1.9), is also unique. Therefore, it is optimal in the sense of minimizing the functional (1.11).

The calculated value of the functional (1.11) in nondimensional units is $Q_1 = 0.029$. By converting to dimensional variables using the corresponding formulas (1.6), we obtain the amount of thermal losses $Q = 33.6\,\mathrm{J}$ (1.5), corresponding to the implemented optimal control.

Thus, the proposed algorithm ensures the construction of an optimal control according to criterion (1.11), satisfying the constraints and providing a high – precision implementation of the required transition.

Conclusion. This study addresses the problem of constructing energy-optimal control for a single-link electromechanical manipulator subject to a control voltage constraint. Based on Pontryagin's maximum principle, it is established that for large transition times, the optimal control corresponds to the unconstrained solution, whereas for small transition times, a combined control structure is required. An algorithm for such control is developed, and its feasibility is confirmed by numerical simulations demonstrating compliance with the control constraint and boundary conditions while ensuring minimal energy consumption.

References

1. Chernousko F. L., Bolotnik N. N., Gradetsky V. G. Manipulation Robots: Dynamics, Control and Optimization. – Boca Raton: CRC Press, 1994. 268 p.
2. Bolotnik N.N., Chernous'ko F.L. Optimization of manipulation robot control // Soviet Journal of Computer Systems Sciences. 1990. Vol. 28. Issue 5. P. 127–169.
3. Avetisyan V.V., Bolotnik N.N. Suboptimal control of an electromechanical manipulator with a high degree of positioning accuracy // Mechanics of Solids. 1990. Vol. 25. Issue 5. P.32-41.
4. Khalil W., Dombre E. Modeling, Identification and Control of Robots. – CRC Press. 2002, 480 p.
5. Spong, M. W., Hutchinson, S., Vidyasagar, M. Robot Modeling and Control. – John Wiley & Sons. 2006, 496 p.
6. Boldea, I., Nasar, S. A. Electric Drives. – Boca Raton: CRC Press. 2016. 672 p.
7. Gregory J., Olivares A., Staffetti E. Energy – optimal trajectory planning for robot manipulators with holonomic constraints // Systems and Control Letters. 2012. Vol. 61. Issue 2. P. 279–291. DOI: 10.1016/j.sysconle.2011.11.005
8. Mirro, J. et al., Optimal control of a single – link flexible manipulator // Journal of Intelligent and Robotic Systems. 1989. Vol. 2. P.187–199.
9. Hussain, S. et al. A robust and energy – efficient trajectory planning framework for high degree of freedom robots //arXiv:2503.11716(2025). DOI: 10.48550/arXiv.2503.11716
10. Demydyuk M.V., Hoshovs'ka N.V. Parametric Optimization of the Transport Operations of a Two – Link manipulator // Journal of Mathematical Sciences. 2019. Vol. 238. No. 2. P. 174-188. DOI: 10.1007/s10958-019-04227-8
11. Oršanský P., Ftorek B., Vittek J. Energy optimal trajectories for electro – mechanical systems // Conference Series: Materials Science and Engineering. 2020. 776(1)012113. DOI: 10.1088/1757-899X/776/1/012113
12. Galicki M. Energy optimal control of mobile manipulators subject to compensation of external disturbance forces // Mechanism and Machine Theory. 2022. Vol. 167. Article 104550. DOI: 10.1016/j.mechmachtheory.2021.104550
13. Abdullah Mohammed, Bernard Schmidt, Lihui Wang, Liang Gao. Minimizing Energy Consumption for Robot Arm Movement // 8th International Conference on Digital Enterprise Technology – Procedia CIRP 25 (2014). P. 400–405. DOI:10.1016/j.procir.2014.10.055
14. Avetisyan V.V. Movements of an electromechanical manipulator robot optimal with respect to energy expenditure // Journal of Computer and Systems Sciences International. 1996. Vol. 35. Issue 4. P. 679–6856.

15. Avetisyan V.V., Martirosyan, S.R. Guaranteed Search for a Target Object by an Electromechanical System with Minimal Light Power Costs // Journal of Computer and Systems Sciences International. 2009. Vol. 48. Issue 5. P. 814–826. DOI: 10.1134/S1064230709050153
16. Avetisyan V.V. Controlled dynamic search for a mobile object with minimum cost of light energy // Automation and Remote Control. 2020. Vol. 81. Issue 4. P. 662–669. DOI:10.1134/S0005117920040086
17. Bolotnik N.N., Sattar T.P. Adaptive time – optimal control of a robotic manipulator // Journal of Computer and Systems Sciences International. 1993. Vol. 31. Issue 3. P. 171–182.
18. Avetisyan V.V., Akulenko L.D., Bolotnik N.N. Optimization of control modes of manipulation robots with regard of the energy consumption // Soviet Journal of Computer Systems Sciences. 1987. Vol. 25. Issue 3. P. 100-107.
19. Pontryagin L.S. et al. Mathematical Theory of Optimal Processes. – M.: Nauka, 1983. 393p. (in Russian).
20. Avetisyan V.V. Optimal Control of an Electromechanical Manipulator with Minimal Energy Consumption // Mechanics – Proceedings of NAS PA. 2023. Vol. 77. Issue 3. P. 58 – 73. (in Russian). DOI: 10.54503/0002-3051-2024.77.3-58
21. Avetisyan V.V., Akulenko L.D., Bolotnik N.N. Modeling and optimization of transport motion for an industrial robot // Soviet Journal of Computer Systems Sciences. 1986. Vol. 24. Issue 6. P. 97 – 103.

Information about author:

Avetisyan Vahan – Doctor of Science (Physics and Mathematics), professor, Institute of Mechanics of NAS of Armenia, тel.: +374 94 449560, e-mail: vanavet@yahoo.com

Received 14 August 2025