ՀԱՅԱՍՏԱՆԻ ԳԻՏՈՒԹՅՈՒՆՆԵՐԻ ԱԶԳԱՅԻՆ ԱԿԱԴԵՄԻԱՅԻ ՏԵՂԵԿԱԳԻՐ
ИЗВЕСТИЯ НАЦИОНАЛЬНОЙ АКАДЕМИИ НАУК АРМЕНИИ

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STABILITY OF AN AXIALLY LOADED BI-MATERIAL BEAM WITH INTERIOR HINGE RESTRAINED BY ELASTIC STRING

Ghazaryan K.B.

Keywords: bi-material beam, stability, interior hinge, elastic support

Ղազարյան Կ.Բ.

Առանցքային բեռնավորմամբ և առաձգական զսպանակով պահվող ներքին հոդակապ ունեցող երկբաղադրիչ հեծանի կայունությունը

Բանալի բառեր. երկու նյութից կազմված հեծան, կայունություն, ներքին հոդակապ, առաձգական հենարան:

Հոդվածում դիտարկվում է առանցքային բեռնավորմամբ և առաձգական զսպանակային հենարանով պահվող ներքին հոդակապ ունեցող երկբաղադրիչ հեծանի խնդիրը: Հեծանի ծայրերում եզրային պայմանների չորս տարբեր կոմբինացիաների համար ստացվել են առանցքային սեղմման ուժի կրիտիկական արժեքները որոշող բնութագրիչ հավասարումները: Թվային վերլուծությամբ ուսումնասիրվել է կրիտիկական առանցքային ուժի կախվածությունը զսպանակային հենարանի կոշտությունից: Ցույց է տրված, որ ներքին հոդակապի հենարանի կոշտությունը զգալիորեն մեծացնում է սեղմող ուժի կրիտիկական արժեքները:

Казарян К.Б.

Устойчивость аксиально нагруженной двухкомпонентной балки с внутренним шарниром, удерживаемым упругой пружиной

Ключевые слова: балка из двух материалов, устойчивость, внутренний шарнир, упругая опора

В статье рассматривается проблема устойчивости двухкомпонентной аксиально нагруженной балки с внутренним шарниром, удерживаемым упругой пружинной опорой. Для четырех различных комбинаций граничных условий на концах балки получены характеристические уравнения, определяющие критические значения силы осевого сжатия. На основе численного анализа изучены зависимости критической осевой силы от жесткости пружинной опоры. Показано, что жесткость опоры внутреннего шарнира значительно увеличивает критические значения сжимающей силы.

The paper discusses a stability problem for a bi-material axially loaded beam with an interior hinge restrained by an elastic spring support. For different four combinations of beam end conditions the characterizing equations are obtained defining critical values of axial compression force. Based on numerical analysis the dependence of critical axial force are studied versus the stiffnesses of spring support. It is shown that the stiffness of the hinge spring support significantly increases beam axial force critical values.

Introduction

The study of stability problems of beams systems is very important in structural design as they are the cornerstone for many structures. Due to technological needs homogeneous and non-homogeneous beams with interior hinges widely used in bolting, swivel designs and suspension bridges. In this paper we specifically deal with static stability of compressed bi-material beams with interior hinge strengthened by elastically constraint supports. The main results of buckling of beams have been given in [1,2]. The stability of beams and plates with hinges are considered in [3-6]. The static stability problem of a compressed inhomogeneous infinite beam with periodically arranged supports is considered in [7]. The stability of multi span beams rested on rigid and elastic supports is studied in [8]. Free vibrations of beams with internal hinges and intermediate translational restraints are investigated in [9-11]. Forced vibrations of finite length meta beams with periodically arranged internal hinges are discussed in [12].

Statements of the problem, the basic equations and contact conditions.

In dimensionless Cartesian coordinates $x = z/L$ were all lengths are normalized to a beam length L we consider a stabilty of a bi- material beam with an interior hinge restrained by an elastic spring support (Fig.1,2) located at point $x = \beta$ $(0 < \beta < 1)$. The beam is compressed by the axial force P applied at the ends of the beam. The four different combinations of beam conditions at the end points $x = 0, x = 1$ will be considered: beam both ends clamped, both ends pinned, one end clamped other end pinned , one end clamped other end sliding.

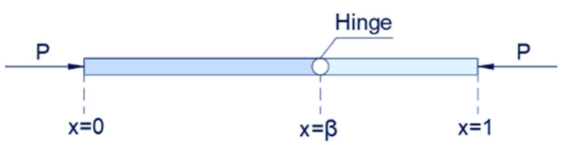

Fig.1 Axially loaded bi-material beam with elastically constraint hinge

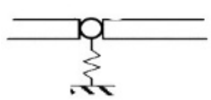

Fig.*2 The elastically constraint hinge*

In dimensionless Cartesian coordinates bi-material beam stability equations can be cast as

$$D_1 \frac{d^4 W_1}{dx^4} + PL^2 \frac{d^2 W_1}{dx^2} = 0; \qquad 0 < x < \beta$$
$$D_2 \frac{d^4 W_2}{dx^4} + PL^2 \frac{d^2 W_2}{dx^2} = 0; \qquad \beta < x < 1 \tag{1}$$

Here $W_j(x)$ is the lateral deflections of the beam neutral axis at point x, D_j, are the flexural rigidities , $j = 1, 2$.

Solutions of the equations (1) are

$$W_1(x) = C_1 \sin(p_1 x) + C_2 \cos(p_1 x) + C_3 x + C_4$$
$$W_2(x) = A_1 \sin(p_2 x) + A_2 \cos(p_2 x) + A_3 x + A_4 \tag{2}$$

where $p_j = \sqrt{D_j^{-1} P L^2}$

The hinge is strengthened by linear elastic restraint with spring constant c_0.

The conditions at the hinge location point $x = \beta$ are:

moments are equal to zero

$$\frac{d^2 W_1(\beta)}{dx^2} = 0, \frac{d^2 W_2(\beta)}{dx^2} = 0 \tag{3}$$

continuity of displacements

$$W_1(\beta) = W_2(\beta) \tag{4}$$

and the balance of shear forces for the compressed beam

$$D_1 \frac{d^3 W_1(\beta)}{dx^3} + PL^2 \frac{dW_1(\beta)}{dx} - D_2 \frac{d^3 W_2(\beta)}{dx^3} - PL^2 \frac{dW_2(\beta)}{dx} = c_0 L^3 W_1(\beta) \tag{5}$$

The solutions satisfying conditions (3-5) can be cast as

$$W_1(x) = C_1 \sin\left(p(x-\beta)\right) + C_3(x-\beta) + C_4$$
$$W_2(x) = A_1 \sin\left(p\gamma(x-\beta)\right) + \left(C_3 - \frac{C_4 \eta}{p^2 \gamma^2}\right)(x-\beta) + C_4 \tag{6}$$

were $\gamma = \sqrt{D_1 D_2^{-1}}$, $\eta = c_0 D_2^{-1} L^3$ is the non-dimensional spring constant , $p = \sqrt{P_0}$, $P_0 = PL^2 D_1^{-1}$ is the non-dimensional normalized axial compressive force .

When the hinge is absent instead of contact conditions (5) we have the conditions of continuities of displacements, slopes, moments and the balance of shear forces

$$W_1(\beta) = W_2(\beta)$$

$$\frac{dW_1(\beta)}{dx} - \frac{dW_2(\beta)}{dx} = 0 \tag{7}$$

$$D_1 \frac{d^2 W_1(\beta)}{dx^2} - D_2 \frac{d^2 W_2(\beta)}{dx^2} = 0$$

$$D_1 \frac{d^3 W_1(\beta)}{dx^3} + PL^2 \frac{dW_1(\beta)}{dx} - D_2 \frac{d^3 W_2(\beta)}{dx^3} - PL^2 \frac{dW_2(\beta)}{dx} = 0 \tag{8}$$

Solutions satisfying these conditions can be written as

$$\begin{aligned} W_1(x) &= C_1 \sin\left(p(x-\beta)\right) + C_2 \cos\left(p(x-\beta)\right) + C_3\left(x-\beta\right) + C_4 \\ W_2(x) &= \gamma^{-1} C_1 \sin\left(p\gamma(x-\beta)\right) + C_2 \cos\left(p\gamma(x-\beta)\right) + C_3\left(x-\beta\right) + A_4 \end{aligned} \tag{9}$$

Clamped-clamped beam

At clamped ends $x = 0, x = 1$ we have the following boundary conditions

$$W_1(0) = 0, \frac{dW_1(0)}{dx} = 0, W_2(1) = 0, \frac{dW_2(1)}{dx} = 0 \tag{10}$$

Satisfying solutions (9) the boundary conditions (10) from non-triviality of all solutions the characteristic equation defining critical buckling force $p^2 = P_0$ find to be

$$\begin{aligned} &\gamma p \cos\left((\beta-1)\gamma p\right)\left(\sin(\beta p)\left(-\beta\eta + \eta + \gamma^2\left(-p^2\right)\right) + p\cos(\beta p)\left((\beta-1)\beta\eta + \gamma^2 p^2\right)\right) + \\ &+ \sin\left((\beta-1)\gamma p\right)\left(p\cos(\beta p)\left(\gamma^2 p^2 - \beta\eta\right) + \eta \sin(\beta p)\right) = 0 \end{aligned} \tag{11}$$

When the hinge is absent instead of (10), using solutions (7) we have the following characteristic equation for the bi-material beam without the interior hinge

$$\begin{aligned} &\sin\left(\beta-1)\gamma p\right)\left(\left(\gamma^2+1\right)\sin(\beta p) - \gamma^2 p\cos(\beta p)\right) + \\ &+ \gamma(\cos\left(\beta-1)\gamma p\right)\left(p\sin(\beta p) + 2\cos(\beta p)\right) - 2) = 0 \end{aligned} \tag{12}$$

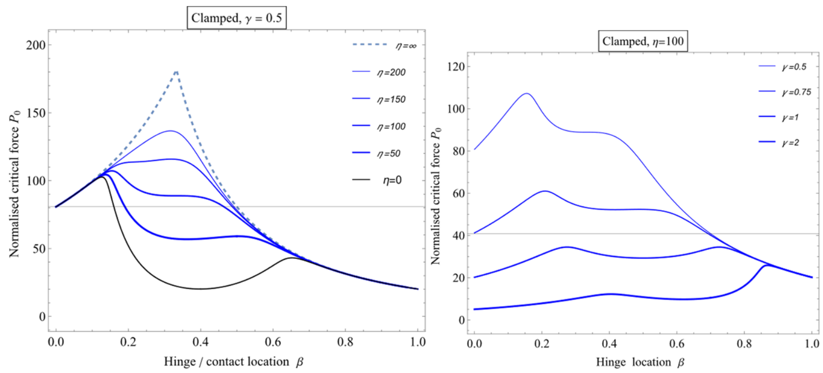

Fig.3a,3b. Critical buckling force P_0 *versus* β *for bi-material clamped beam*

On the Fig.3a the curves of normalized critical buckling force $P_0 = q^2$ corresponding to the smallest roots of the equation (11) dependence functions from hinge location parameter β are given for non- homogeneous beams $\gamma = 0.5$ versus for different values of spring stiffnesses η. Black curve corresponds to case when the hinge is not support. On Fig.3b the critical curves are plotted for non- homogeneous beams with different values of γ when the hinge spring stiffnesses $\eta = 100$. The optimum locations of the hinge correspond to points where critical forces are maximal. For comparison on the Fig. 4 the plots of normalized buckling force $P_0 = p^2$ versus contact location point for bi-material clamped beam in when interior hinge is absent.

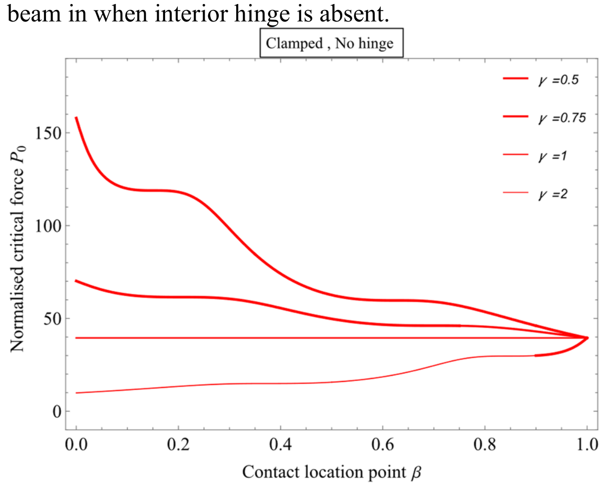

Fig. 4 Critical buckling force P_0 *versus* β *for clamped bi-material beam without hinge*

Pinned-pinned beam

At the pinned ends we have the following boundary conditions

$$W_1(0) = 0, \frac{d^2 W_1(0)}{dx^2} = 0, \qquad W_2(1) = 0, \frac{d^2 W_2(1)}{dx^2} = 0 \tag{13}$$

The characteristic equation corresponding to these conditions can be founded as

$$\left(\beta\eta(1-\beta)-\gamma^2 p^2\right)\sin\left(p\gamma(1-\beta)\right)\sin\left(p\beta\right)=0 \qquad (14)$$

From (12) it follows that, when only the left uncoupled part of the beam buckles $P_0=\pi^2/\beta^2$, when only the right uncoupled part buckles $P_0=\pi^2/\left(\gamma(1-\beta)\right)^2$.

When both parts buckle together $P_0=\beta\eta(1-\beta)/\gamma^2$.

For beam without hinge the characteristic equation corresponding to conditions (11) can be written as

$$\gamma\sin(\beta p)\cos\left((\beta-1)\gamma p\right)+\cos(\beta p)\sin\left((1-\beta)\gamma p\right)=0 \qquad (15)$$

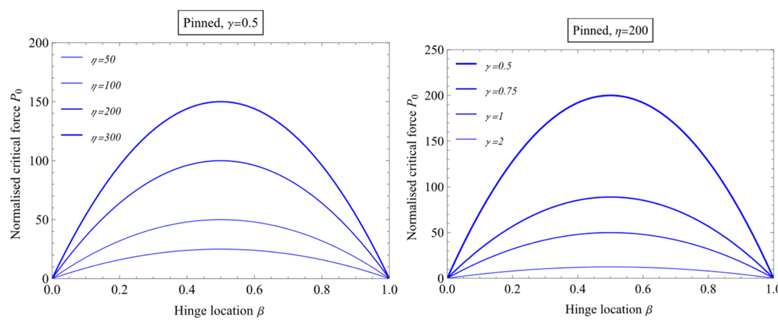

Fig.5a,5b. Critical buckling force P_0 *versus* β *for bi-material pinned beam*

The hinge optimum location where the critical buckling forces are maximal is at the midpoint $\beta=0.5$

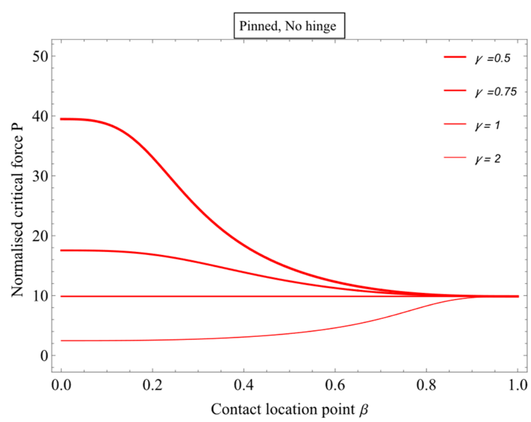

Fig. 6 Critical buckling force P_0 *versus* β *for pinned bi-material beam without hinge*

Clamped-pinned beam

At the one clamped and other pinned ends we have the following boundary conditions

$$W_1(0)=0, \frac{dW_1(0)}{dx^2}=0,\ W_2(1)=0, \frac{d^2W_2(1)}{dx^2}=0 \tag{16}$$

Corresponding to these conditions the characteristic equation defining critical force can be found as

$$\begin{aligned}&\gamma^2 p^2 \sin\left((1-\beta)\gamma p\right)\left(p\cos(\beta p)-\sin(\beta p)\right)+\\&+(\beta-1)\eta\sin\left((1-\beta)\gamma p\right)\left(\beta p\cos(\beta p)-\sin(\beta p)\right)=0\end{aligned} \tag{17}$$

For beam without hinge the characteristic equation can be cast as

$$\begin{aligned}&\gamma p^2(\gamma\cos\left(\beta-1)\gamma p\right)\left(p\cos(\beta p)-\sin(\beta p)\right)+\\&+\sin\left(\beta-1)\gamma p\right)\left(p\sin(\beta p)+\cos(\beta p)\right)=0\end{aligned} \tag{18}$$

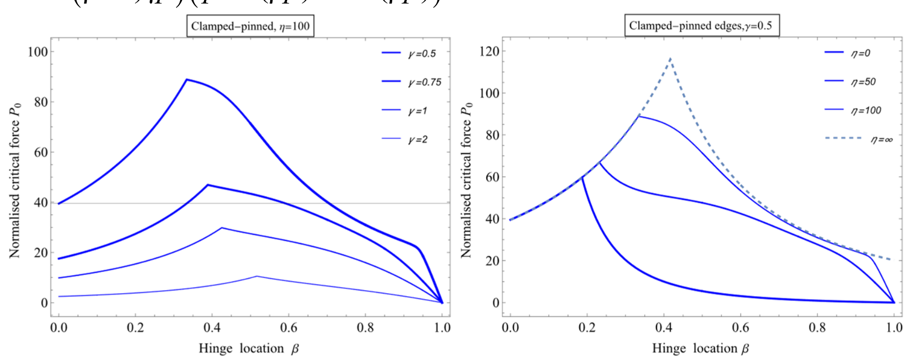

Fig.7a,7b. Critical buckling force P_0 *versus* β *for bi-material clamped-pinned beam*

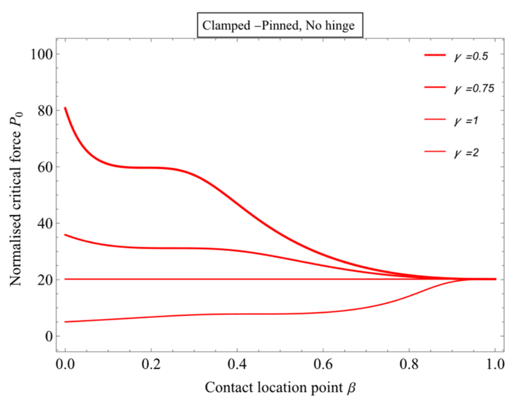

Fig. 8 Critical buckling force P_0 *versus* β *for clamped-pinned bi-material beam without hinge*

Clamped -sliding beam

At the one clamped and other sliding ends we have the following boundary conditions

$$W_1(0)=0, \frac{dW_1(0)}{dx}=0, \ \frac{dW_2(1)}{dx}=0, \frac{d^3W_2(1)}{dx^3}=0 \tag{19}$$

The characteristic equation defining critical force can be found as

$$\left(\gamma q\left((1-\beta)\eta-\gamma^2 q^2\right)\cos\left((1-\beta)\gamma q\right)-\eta\sin\left((1-\beta)\gamma q\right)\right)q\cos(\beta q)=0 \tag{20}$$

For beam without hinge we have the following characteristic equation defining critical force

$$\sin(\beta p)\cos\left(\beta-1\right)\gamma p\right)+\gamma\cos(\beta p)\sin\left(\beta-1\right)\gamma p\right)=0 \tag{21}$$

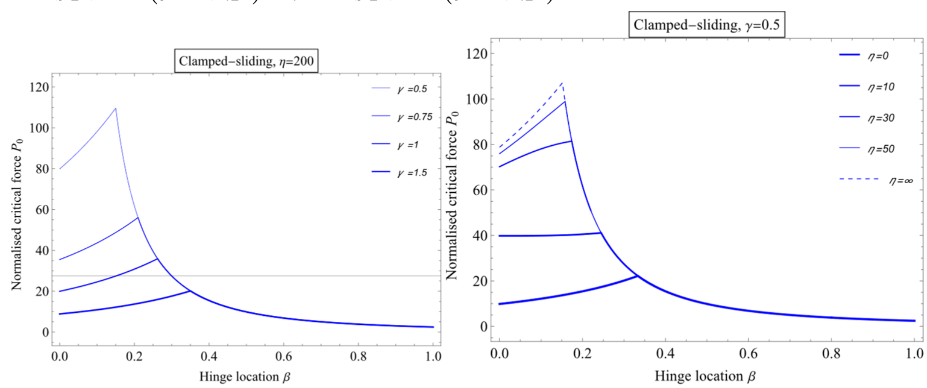

Fig9a,9b. Critical buckling force P_0 *versus* β *for bi-material clamped-sliding beam*

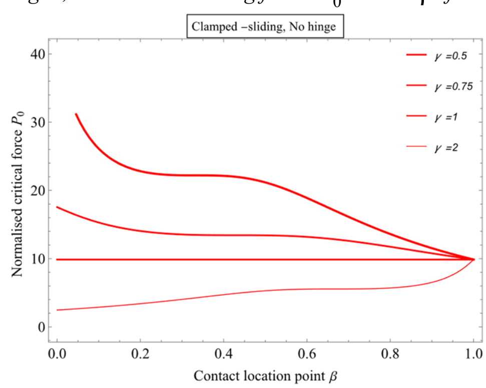

Fig. 10 Critical buckling force P_0 *versus* β *for clamped- sliding bi-material beam without hinge*

Results

As results from plots of Figures 3-10 for all considered boundary value problems the interior hinge weakens the bi-material beam in stability decreasing critical values of compressed force.

The stiffness of the hinge elastic support makes the beam more stable, increasing critical values of compressed force.
The critical values of the axial force for the bi-material beam with restrained interior hinge and the bi-material beam without hinge is mostly similar for sufficiently large values of the spring stiffnesses in all considered cases.

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Information about autor
Karen Ghazaryan Department of dynamics of deformable systems and coupled fields, Institute of mechanics of NAS of Armenia, Yerevan,E-mail: ghkarren@gmail.com

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