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ИЗВЕСТИЯ НАЦИОНАЛЬНОЙ АКАДЕМИИ НАУК АРМЕНИИ

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LOCALISED VIBRATIONS OF HOMOGENEOUS STRING WITH FINITE NUMBER OF PERIODICALLY LOCATED SCATTERERS

Ghazaryan K., Ghazaryan R., Terzyan S.

Keywords: periodic structure, localization, string vibration, imperfect elastic contact

Казарян К.Б., Казарян Р.А., Терзян С.А.
Локализованные колебания однородной струны с конечным числом периодически расположенных рассеивателей

Ключевые слова: периодическая структура. локализация, колебание струны, несовершенный упругий контакт.

Рассмотрена задача локализации волн напряжений в однородной растянутой струне с закрепленными концами. В струне имеются точки несовершенного упругого контакта, обусловленного рассеивателями, периодически распределенными по длине струны. Показано, что в этой периодической структуре возникает локализованная волна напряжения, вызванная рассеивателями.

Ղազարյան Կ.Բ., Ղազարյան Ռ.Ա., Թերզյան Ս.Հ.
Վերջավոր թվով պարբերաբար դասավորված ցրիչներով համասեռ լարի տեղայնացված տատանումները

Հիմնաբառեր՝. Պարբերական կառուցվածք, տեղայնացում, լարի տատանումներ, թերի առաձգական կոնտակտ

Դիտարկված է ամրակցված ծայրերով համասեռ ձգված լարի լարման ալիքների տեղայնացման խնդիրը: Լարում առկա են թերի առաձգական կոնտակտի կետեր, պայմանավորված լարի երկարությամբ պարբերաբար դասավորված ցրիչներով: Ցույց է տրված, որ այս պարբերական կառուցվածքում առաջանում է ցրիչներով պայմանավորված լարման տեղայնացված ալիք:

The problem of localisation of stress waves is considered in homogeneous fixed string in tension with interfaces of imperfect elastic contact caused by scatterers periodically oriented along string length. It is shown that in this periodic structure due to scatterers the localisation of stress wave is occur.

Introduction

In the paper based on the propagator matrix formalism in conjunction of Sylvester's matrix theorem localized vibration of a fixed string in tension is studied. The string contains finite number non equidistant located scaterrers (micro inhomogeneities, point masses, beads) periodically oriented along string length. At points where scatterers are located the stress traction discontinuity is taken to be linearly related to the continuous displacement.

The dynamics of elastic, electro-elastic waves in one dimensional periodic structures is a well-studied classic topic [1-3]. Gap bands, localisation, attenuation, reflection, refraction, resonance and other effects in finite, semi-infinite and infinite structures have been investigated by many researchers, particularly in [4-8]. The vibration of periodic strings and rod with scatterers, local resonators are studied in [9-13]. In elastic and electro- elastic structures models of an imperfectly bonded interfaces and problems based on these models are proposed and studied in [14-22].

Statement and solution of the problem

Consider a string under tension T_0 with finite number of scatterers (micro inhomogeneities, point masses, beads) periodically oriented along length of string at points $x=(n-1)d+d_1, x=nd,\ n=1,2..N, d=d_1+d_2$. (Fig.1)

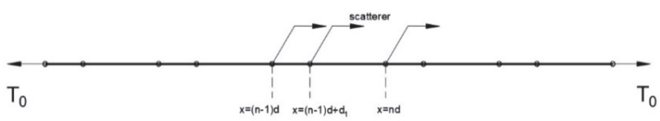


Fig.1 String under tension with periodically oriented scatterers

We have the following equation of a string transverse vibration

$$T_0\frac{\partial^2 V}{\partial x^2}+\rho\frac{\partial^2 V}{\partial t^2}=0; \tag{1}$$

where T_0 is the tension , ρ is the mass density per unit length of string .

Considering a harmonic vibration (ω is the frequency)

$$V(x,t)=U(x)\exp\left(i\omega t\right)$$

and introducing in the repeated unit cell $x\in\left((n-1)d, nd\right)$ (Fig.1) the vectors

$$\mathbf{U}_n^{(j)}(x)=\begin{pmatrix}U_n^{(j)}(x)\\ \sigma_n^{(j)}(x)\end{pmatrix}, \mathbf{C}_n^{(j)}=\begin{pmatrix}A_n^{(j)}\\ B_n^{(j)}\end{pmatrix}$$

$$\sigma_n^{(j)}(x)=T_0\frac{dU_n^{(j)}(x)}{dx}, \tag{2}$$

the solutions of (1) within each sub-sells can be cast as

$$\mathbf{U}_n^{(j)}(x)=\mathbf{P}(x)\mathbf{C}_n^{(j)}$$

$$\mathbf{P}(x)=\begin{pmatrix}\sin(qx) & \cos(qx)\\ T_0 q\cos(qx) & -T_0 q\sin(qx)\end{pmatrix}; \qquad q=\omega\sqrt{\frac{\rho}{T_0}}; \tag{3}$$

Here $A_n^{(j)}, B_n^{(j)}$ are constants, n is the number of the unit cell, the indexes $j=1,2$ stand for the sub-sells, $x \in \left((n-1)d, (n-1)d+d_1\right)$, $x \in \left((n-1)d+d_1, nd\right)$, $d = d_1 + d_2$ respectively .

Propagator matrix approach

Based on the procedure of propagator matrix approach [23] considering any two neighbouring points x_1, x_2 of sub-cells we can construct the transfer matrix $\mathbf{T}$ in such way that

$$\mathbf{U}_{nj}(x_2) = \mathbf{T}\left(x_2 - x_1\right)\mathbf{U}_{nj}(x_1) \tag{4}$$

where

$$\begin{aligned} &\mathbf{T}\left(x_2 - x_1\right) = \mathbf{P}(x_2)\mathbf{P}^{-1}(x_1) \\ &\mathbf{T}\left(x_2 - x_1\right) = \begin{pmatrix} \cos\left(q\left(x_2 - x_1\right)\right) & \left(T_0 q\right)^{-1}\sin\left(q\left(x_2 - x_1\right)\right) \\ -T_0 q \sin\left(q\left(x_2 - x_1\right)\right) & \cos\left(q\left(x_2 - x_1\right)\right) \end{pmatrix} \end{aligned} \tag{5}$$

Using (5) the following relations can be obtained

$$\begin{aligned} &\mathbf{U}_n^{(1)}\left((n-1)d + d_1\right) = \mathbf{T}\left(d_1\right)\mathbf{U}_n^{(1)}\left((n-1)d\right) \\ &\mathbf{U}_n^{(2)}\left(nd\right) = \mathbf{T}\left(d_2\right)\mathbf{U}_n^{(2)}\left((n-1)d + d_1\right) \end{aligned} \tag{6}$$

Here

$$\mathbf{T}\left(d_j\right) = \begin{pmatrix} \cos\left(qd_j\right), & (T_0 q)^{-1}\sin\left(qd_j\right) \\ -T_0 q \sin\left(qd_j\right), & \cos\left(qd_j\right) \end{pmatrix} \tag{7}$$

At points $x = \left(d_1 + d(n-1)\right), x = nd$ were scatterers are located we take the following imperfect contact conditions, $f \geq 0,$ [11]

$$\begin{aligned} &U_n^{(2)}\left(d_1 + d(n-1)\right) - U_n^{(1)}\left(d_1 + d(n-1)\right) = 0, \\ &\sigma_n^{(2)}\left(d_1 + d(n-1)\right) - \sigma_n^{(1)}\left(d_1 + d(n-1)\right) = fU_n^{(1)}\left(d_1 + d(n-1)\right) \end{aligned} \tag{8}$$

$$\begin{aligned} &U_{n+1}^{(1)}\left(nd\right) - U_n^{(2)}\left(nd\right) = 0 \\ &\sigma_{n+1}^{(1)}\left(nd\right) - \sigma_n^{(2)} nd = fU_n^{(2)}\left(nd\right) \end{aligned} \tag{9}$$

In matrix form these conditions can be written as

$$\mathbf{U}_n^{(2)}\left(d(n-1+d_1)\right)=\mathbf{FU}_n^{(1)}\left(d(n-1)+d_1\right)$$
$$\mathbf{U}_{n+1}^{(1)}\left(nd\right)=\mathbf{FU}_n^{(2)}\left(nd\right) \tag{10}$$

where

$$.\ \mathbf{F}=\begin{pmatrix}1 & 0\\ f & 1\end{pmatrix}$$

Taking into account (8 -10) we can obtain the following relation

$$\mathbf{U}_{n+1}^{(1)}\left(nd\right)=\mathbf{FU}_n^{(2)}\left(nd\right)=\mathbf{FT}\left(d_2\right)\mathbf{U}_n^{(2)}\left((n-1)d+d_1\right)=$$
$$=\mathbf{FT}\left(d_2\right)\mathbf{FU}_n^{(1)}\left((n-1)d+d_1\right)=\mathbf{FT}\left(d_2\right)\mathbf{FT}\left(d_1\right)\mathbf{U}_n^{(1)}\left((n-1)d\right)$$
$$\mathbf{U}_{n+1}^{(1)}\left(nd\right)=\mathbf{MU}_n^{(1)}\left((n-1)d\right) \tag{11}$$

Here

$$\mathbf{M}=\mathbf{FT}\left(d_2\right)\mathbf{FT}\left(d_1\right),\ \ \mathbf{M}=\begin{pmatrix}m_{11} & m_{12}\\ m_{21} & m_{22}\end{pmatrix}$$

is the unimodal propagator matrix, elements of which can be cast as

$$\begin{aligned}
m_{11}&=\frac{\xi\sin(\beta p)\cos(p-\beta p)}{p}+\cos(p);\\
m_{12}&=\frac{d(\xi\sin(\beta p)\sin(p-\beta p)+p\sin(p))}{T_0p^2}\\
m_{21}&=\frac{T_0\left(-2p^2\sin(p)-\xi^2\sin(p-2\beta p)+\xi p\cos(p-2\beta p)+\xi^2\sin(p)+3\xi p\cos(p)\right)}{2dp}\\
m_{22}&=\frac{\left(2p^2-\xi^2\right)\cos(p)+\xi(\xi\cos(p-2\beta p)+p(\sin(p-2\beta p)+3\sin(p)))}{2p^2}
\end{aligned} \tag{12}$$

In (12) $p=qd=\Omega$, $\Omega=\omega d\sqrt{\rho/T_0}$ is the non-dimensional frequency, $\xi=dfT_0^{-1}$ is the dimensionless “scattering” parameter and $\beta=d_2/d$ is the relative distance parameter.

Since the vectors $\mathbf{U}\left(nd\right)$ are continuous at the interface points of the neighbouring cells repeating relations (11) the n-th times the matrix $\mathbf{M}^n$ can be found .

The matrix $\mathbf{M}^n$ for any $n=1,2,..N$ links the values of field vectors at $x=0$ and $x=nd$ points of the string

$$\mathbf{U}_n\left(nd\right)=\mathbf{M}^n\mathbf{U}_0\left(0\right) \tag{13}$$

According to Sylvester's matrix theorem [24] for 2×2 matrix the elements of the n -th power of the matrix $\mathbf{M}^n$ can be cast as

$$\mathbf{M}^n = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$$

and can be simplified using the following matrix identities

$$\begin{aligned} &M_{11} = m_{11}S_{n-1}(\theta) - S_{n-2}(\theta); \quad M_{12} = m_{12}S_{n-1}(\theta) \\ &M_{21} = m_{21}S_{n-1}(\theta); \qquad M_{22} = m_{22}S_{n-1}(\theta) - S_{n-2}(\theta) \end{aligned} \tag{14}$$

where $S_n(\theta)$ are the Chebyshev polynomials of second kind

$$S_n(\theta) = \frac{\sin\left((n+1)\arccos(\theta)\right)}{\sin\left(\arccos(\theta)\right)}; \qquad \theta = \frac{1}{2}\operatorname{tr}(\mathbf{M}) \tag{15}$$

Note that function $\theta(\Omega)$ defines the band gaps structure in infinite string [7]

$$\cos(kd) = \theta(\Omega) \tag{16}$$

where k is the Bloch wave number.

The relation establishing a link between values of the vectors $\mathbf{U}(Nd)=\mathbf{M}^N\mathbf{F}^{-1}\mathbf{U}(0)$ will enable to consider the boundary value problem of a string free vibration with fixed ends.

$$U_0(0) = U_N(Nd) = 0 \tag{17}$$

From (13) and (17) it follows that

$$\begin{pmatrix} 0 \\ \sigma_N(Nd) \end{pmatrix} = \mathbf{M}^N\mathbf{F}^{-1}\begin{pmatrix} 0 \\ \sigma_0(0) \end{pmatrix} \tag{18}$$

Here $\sigma_0(0), \sigma_N(Nd)$ are stresses arising at the string fixed ends.

Taking into account that

$$\mathbf{F}^{-1} = \begin{pmatrix} 1 & 0 \\ -f & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 \\ -f & 1 \end{pmatrix}\begin{pmatrix} 0 \\ \sigma(0) \end{pmatrix} = \begin{pmatrix} 0 \\ \sigma(0) \end{pmatrix}$$

from matrix equation (18) one can obtain

$$m_{12}(\Omega)S_{N-1}(\theta)\sigma(0) = 0 \tag{19}$$

$$\sigma_N\left(Nd\right)=\left(m_{22}(\Omega)S_{N-1}\left(\theta\right)-S_{N-2}\left(\theta\right)\right)\sigma_0\left(0\right) \tag{20}$$

Equation (19), $\left(\sigma(0)\neq 0\right)$ gives two alternative families of normal and localised vibrational modes [6]

$$m_{12}(\Omega)=0 \tag{21}$$

$$S_{N-1}\left(\theta\right)=0 \tag{22}$$

Equation (21) defines the countable set solutions of localised wave frequencies

$\Omega_i, i=1,2,....\infty$.

At frequencies Ω_i, $m_{22}(\Omega_i)m_{11}(\Omega_i)=1$ and therefore

$$\theta=\frac{m_{11}(\Omega_i)+m_{22}(\Omega_i)}{2}=\frac{\lambda+\lambda^{-1}}{2}, \text{ where } \lambda=m_{22}(\Omega_i) \tag{23}$$

Using the recurrence formula for Chebyshev polynomials the following relation can be obtained [6]

$$\sigma_n\left(nd\right)=\lambda^n\sigma_0\left(0\right), n=1,2....N \tag{24}$$

From (24) it follows that if $\left|\lambda(\Omega_j)\right|<1$ the stress wave localisation occur at $x=0$, in the case of $\left|\lambda(\Omega_j)\right|>1$ the stress wave localisation occur at $x=L$.

Another possible case is the equation (21). This equation has $N-1$ roots in the interval $\theta\in\left(-1,1\right)$, which are given by

$$\theta_{0m}=\cos\left(m\pi N^{-1}\right), m=1,2...N-1 \tag{25}$$

Taking into account that $S_{N-2}\left(\theta_{0m}\right)=(-1)^m$ one can write

$$\sigma_0\left(nd\right)=(-1)^m\sigma_0\left(0\right), n=1,2....N \tag{26}$$

This means that $N-1$ normal modes exist where waves are uniformly distributed along the string length.

Analysis and results

It follows from (16) that the condition $\left|\theta\left(\Omega\right)\right|>1$ defines band gaps in an infinite string. Since the imaginary parts of Bloch vector $\mathrm{Im}\left(kd\right)$ operate inside the gaps, the analysis of ban gap structure caused by scatterers will be carry out by considering the attenuation function $\mathrm{Im}\left(kd\right)$ within band gaps.

The influence of the scattering on formation of band gaps is illustrated on Fig. 2 where the imaginary parts (attenuation curves) of the Bloch wave vectors are plotted as a function

of non-dimensional frequency Ω, the lowest contours of the attenuation curves where $\mathrm{Im}\left(kd\right)\to 0$ define the map of band gap frequencies. The maps correspond to the first and second gaps. The curves are plotted at $\beta = 0.3$.
Hereafter the blue curves correspond to scattering coefficient $\xi = 10$, the black curves to $\xi = 5$, the red curves to $\xi = 2$ *(See online version for colors).* Analysis shows that band gap structure slightly depends from β which means that gaps may open also in the case of uniformly (equidistant) oriented scaterrers.

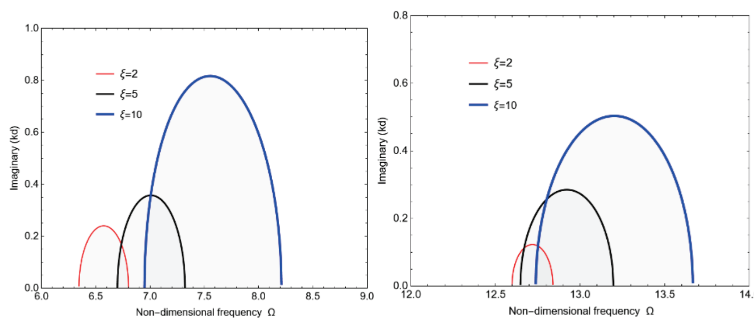
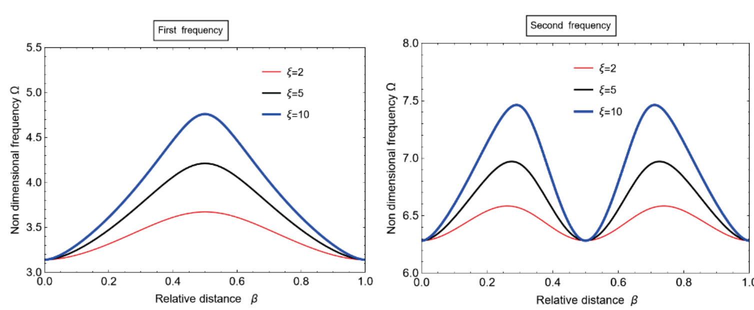


Figure 2. Map of first and second gaps

As it follows from Figure 2. scattering essentially increase the widths of the gaps and the attenuation function within gaps.

The eigen frequencies Ω_i versus relative distance β shown in Figure 3., where Ω_i are the solutions of the equation (21). Note that $\Omega_i\left(\beta\right) = \Omega_i\left(1-\beta\right)$



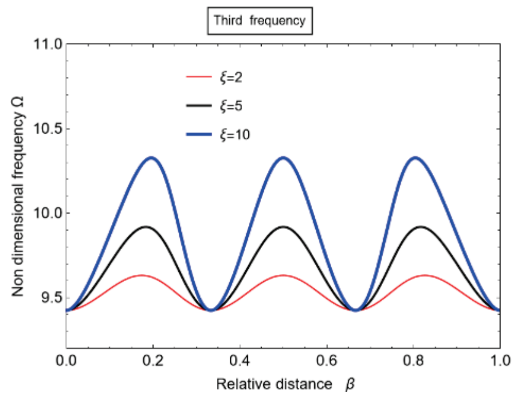


Figure3. The frequencies curves versus relative distance β

As it follows from Figure 3. scattering increase the frequency of localised wave and is more increasing the maximal value of the low frequencies than the maximal value of high frequencies.

On the Figures 4a,b the plots of localisation coefficients versus relative distance β are presented. Note that $\left|\lambda(\Omega_i,\beta)\right| = \left|\lambda\left(\Omega_i(1-\beta)\right)\right|^{-1}$.

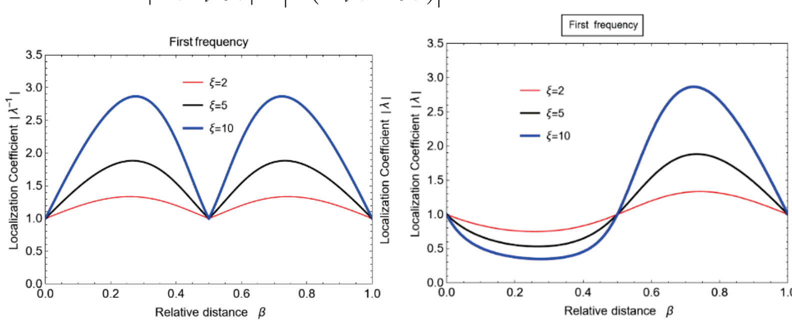


Figure 4a. Graphs of localisation coefficients versus relative distance

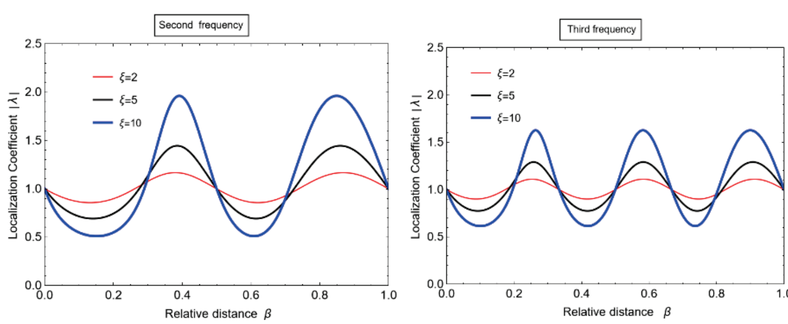


Figure 4b. Graphs of localisation coefficients versus relative distance

As it follows from Figure 4. the scattering sufficiently increases the localisation parameter for low frequencies. From analysis of the graphs of Fig .4 it follows that due to scattering the very strong localisation of the stress wave in the string occur even for $\xi = 2$ and $N = 10$.

When $\xi = 2, \max\left|\lambda(\Omega_3)\right| = 1.62, \left(\min\left|\lambda(\Omega_3)\right| = 0.61\right)$ $\sigma_N(Nd)/\sigma_0(0) \sim 0.0082$.
Note that the localisation effect is stronger at the first frequency.

Conclusions

Based on the transfer matrix procedure in conjunction of Sylvester's matrix theorem the problem of localized wave is studied in a string in tension with interfaces of imperfect elastic contact caused by periodically non equidistant oriented scatterers. It is shown that in this periodic structure the localisation of stress wave occur. The localised stress wave frequencies and wave localisation amplitude depending both scattering factor and relative distance between scaterrers are determined analytically and illustrated by plots. It is shown also that in the infinite string the periodic oriented scatterers can open band gaps.

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Information about authors

Ghazaryan K. Chief researcher, Institute of Mechanics NAN of Armenia, ghkarren@gmail.com, Phone 374 99 227395

Ghazaryan R. Researcher, Institute of mechanics NAN of Armenia, rafaelghazaryan52@gmail.com, 374 99 396344

Terzyan S., Researcher, Institute of mechanics NAN of Armenia, sat_and_21@yahoo.com,

374 99 340432

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