ՀԱՅԱՍՏԱՆԻ ԳԻՏՈՒԹՅՈՒՆՆԵՐԻ ԱԶԳԱՅԻՆ ԱԿԱԴԵՄԻԱՅԻ ՏԵՂԵԿԱԳԻՐ
ИЗВЕСТИЯ НАЦИОНАЛЬНОЙ АКАДЕМИИ НАУК АРМЕНИИ

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Free Vibrations of Thin Elastic Orthotropic Cylindrical Panels with Arbitrary Fastening of the Ends

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Key words: free vibrations, cylindrical panel, rigid-clamped, hinged ends.

Using the system of equations corresponding to the classical theory of orthotropic cylindrical shells, the free vibrations of thin elastic orthotropic cylindrical panels with arbitrary fastening of the ends are investigated. To calculate the natural frequencies and to identify the respective natural modes, the generalized Kantorovich-Vlasov method of reduction to ordinary differential equations is used. Dispersion equations for finding the natural frequencies of possible types of vibrations are derived. An asymptotic relation between the dispersion equations of the problems at hand and the analogous problems for a rectangular plate is established. An algorithm for separating possible vibrations is presented. As examples, the values of dimensionless characteristics of natural frequencies are derived for orthotropic cylindrical panels.

Ծայրերում կամայական ամրացմամբ բարակ առաձգական օրթոտրոպ գլանային վահանակների ազատ տատանումները

Ղուլղազարյան Գ.Ռ., Ղուլղազարյան Լ.Գ.

Հիմնաբառեր: Ազատ տատանումներ, գլանային վահանակ, կոշտ ամրակցված, հոդակապորեն ամրակցված ծայրեր:

Օգտվելով օրթոտրոպ գլանային թաղանթների դասական տեսությանը համապատասխան հավասարումների համակարգից, հետազոտվում է ծայրերում կամայական ամրացմամբ բարակ առաձգական օրթոտրոպ գլանային վահանակների ազատ տատանումները: Սեփական հաճախությունների արժեքները և նրանց համապատասխան սեփական ֆունկցիաները գտնելու համար կիրառվում է սովորական դիֆերենցիալ հավասարումների բերման Կանտորովիչ-Վլասովի մեթոդը: Հնարավոր տիպերի սեփական տատանումների հաճախությունները գտնելու համար արտածված են դիսպերսիոն հավասարումներ: Ասիմպտոտիկ կապ է հաստատված դիտարկվող խնդրների և ուղղանկյուն օրթոտրոպ սալի համար համապատասխան խնդրների դիսպերսիոն հավասարումների միջև: Բերված է մեխանիզմ, որի օգնությամբ կատարվում է հնարավոր տիպի տատանումների տարանջատում: Օրթոտրոպ գլանային վահանակի օրինակների վրա ստացված են սեփական տատանման մոտավոր հաճախությունների անչափողական բնութագրիչները:

Свободные колебания тонкой упругой ортотропной цилиндрической панели с произвольным закреплением торцов

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Ключевые слава: Свободные колебания, цилиндрическая панель, жестко защемленные, шарнирно-закрепленные торцы.

Используя систему уравнений соответствующей классической теории ортотропных цилиндрических оболочек, исследуются свободные колебания ортотропной тонкой упругой цилиндрической панели с произвольным закреплением торцов. Для расчета собственных частот и идентификации соответствующих собственных мод используется обобщенный метод сведения к обыкновенным дифференциальным уравнениям Канторовича-Власова. Получены дисперсионные уравнения для нахождения собственных частот возможных типов колебаний. Установлена асимптотическая связь между дисперсионными уравнениями рассматриваемой задачи и аналогичной задачи для ортотропной прямоугольной пластины.

Приводится механизм, с помощью которого расчленяются возможные типы краевых колебаний. На примерах ортотропной цилиндрической панели получены приближенные значения безразмерной характеристики собственных частот колебаний.

Introduction. It is known that, at the free edge of an orthotropic plate planar and flexural vibrations can occur independently of each other [1-6]. When the plate is bent these vibrations become coupled and giving raise to two new types of vibrations localized at the free edge: predominantly tangential and predominantly bending vibrations. The transformation of the one type of vibration into the other occurs at the free edge of a thin cylindrical elastic panel. For these vibrations a complex distribution of frequencies of natural vibrations occurs depending on the geometrical and mechanical parameters of finite and infinite cylindrical panels [4]-[10]. With the increase of the number of free edges of a cylindrical panel the distribution becomes increasingly complex [7]-[10], [19-21]. Therefore, the investigation of the edge resonance of cylindrical panels with arbitrary fastening of the ends, when other edges are free, is one of the most difficult problems in the theory of vibrations of plates [4]. These difficulties are resolved by using a combination of analytical and asymptotic theories, as well as by numerical methods.

In the present work, for the first time, free vibrations of rectangular plates with arbitrary fastening of the opposite sides and cylindrical panels with arbitrary fastening of the ends are investigated. Such elements are important components of modern structures and constructions. Therefore, the question of free vibrations of these elements is of vital importance and it requires special attention. It is proved that these problems prevent separation of variables for given boundary conditions (except for the hinged ends). It can be proved that such problems for cylindrical panels of orthotropic materials with simple boundary conditions are self-conjugate and nonnegative definite. Therefore, the generalized Kantorovich-Vlasov method can be applied to them [11]-[15]. As the basic functions the following eigenfunctions of the problems are used: a) for cylindrical panel with rigid-clamped ends:

$$w^{IV} = \theta^4 w;\; w(0) = w'(0) = 0, w(l) = w'(l) = 0\;;\; 0 \le \alpha \le l. \quad (1)$$

b) for cylindrical panel with rigid-clamped and hinged end:

$$w^{IV} = \theta^4 w;\; w(0) = w'(0) = 0, w(l) = w''(l) = 0\;;\; 0 \le \alpha \le l. \quad (2)$$

c) for cylindrical panel with hinged ends:

$$w^{IV} = \theta^4 w;\; w(0) = w''(0) = 0, w(l) = w''(l) = 0\;;\; 0 \le \alpha \le l. \quad (3)$$

Note that the function w(α) characterizes the deflection of the beam axis from the equilibrium position [3].

The problems (1)-(3) are self-conjugate and have nonnegative simple discrete spectrums with a limit point at the infinity. The eigenfunctions corresponding to the eigenvalues θ_m^4, m=$\overline{1,\infty}$ of the problems (1)-(3) have the forms, respectively:

$$w_m^1(\theta_m^1\alpha) = sh\left(\frac{\theta_m^1 l}{2}\right)\left(ch(\theta_m^1\alpha) - \cos(\theta_m^1\alpha)\right) - ch\left(\frac{\theta_m^1 l}{2}\right)\left(sh(\theta_m^1\alpha) - \sin(\theta_m^1\alpha)\right), 0 \le \alpha \le l, \text{m}=\overline{1,\infty}. \quad (4)$$

$$w_m^2(\theta_m^2\alpha) = sh(\theta_m^2 l)\left(ch(\theta_m^2\alpha) - \cos(\theta_m^2\alpha)\right) - ch(\theta_m^2 l)\left(sh(\theta_m^2\alpha) - \sin(\theta_m^2\alpha)\right),\quad 0 \le \alpha \le l, \text{m}=\overline{1,\infty}. \quad (5)$$

$$w_m^3(\theta_m^3\alpha) = sin(\theta_m^3\alpha), 0 \le \alpha \le l, \text{m}=\overline{1,\infty}. \quad (6)$$

These eigenfunctions with their first and second derivatives define an orthogonal basis in the Hilbert space $L_2[0,l]$ [15]. Here θ_m^i, i=1,2,3; m=$\overline{1,\infty}$ are positive zeros of the equations:

$$ch(\theta l)\,cos(\theta l) = 1. \qquad (7)$$

$$\tanh(\theta l) = \tan(\theta l). \qquad (8)$$

$$sin(\theta l) = 0. \qquad (9)$$

Denote

$$\beta'^{i}_{m} = \frac{\int_0^l \left(w_m^{i\prime}(\theta_m^i\alpha)\right)^2 d\alpha}{\int_0^l \left(w_m^{i}(\theta_m^i\alpha)\right)^2 d\alpha},\ \beta''^{i}_{m} = \frac{\int_0^l \left(w_m^{i\prime\prime}(\theta_m^i\alpha)\right)^2 d\alpha}{\int_0^l \left(w_m^{i\prime}(\theta_m^i\alpha)\right)^2 d\alpha};\ \text{i=1,2,3}. \qquad (10)$$

Notice that, the derivatives in formulas (10) are taken with respect to $\theta_m^i\alpha$ and

$$\beta'^{i}_{m} \to 1, \beta''^{i}_{m} \to 1\ at\ m \to \infty, i = 1,2;\ \beta'^{3}_{m} = 1, \beta''^{3}_{m} = 1, \mathrm{m} = \overline{1,\infty}\,.$$

1. The Statement of Problems and Basic Equations.

It is assumed that the generatrixes of the cylindrical panels are orthogonal to the ends of the panels. The curvilinear coordinates (α,β) are defined on the median surface of the panels where $\alpha(0 \le \alpha \le l)$ and $\beta(0 \le \beta \le s)$ are the lengths of the generatrix and the directing circumference, respectively; l – is the length of the panels; and s – is the length of the directing circumferences.

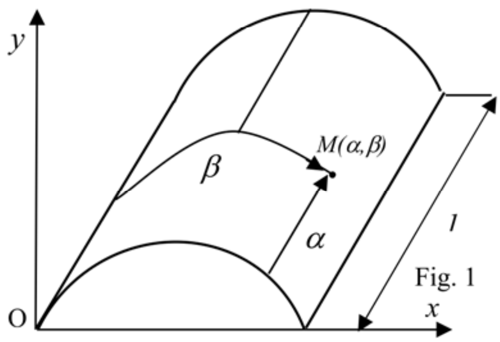

Fig. 1

As the initial equations describing vibrations of the panels, we will use the equations corresponding to the classical theory of orthotropic cylindrical shells written in the selected curvilinear coordinates α and β (Fig. 1):

$$-B_{11}\frac{\partial^2 u_1}{\partial\alpha^2} - B_{66}\frac{\partial^2 u_1}{\partial\beta^2} - (B_{12}+B_{66})\frac{\partial^2 u_2}{\partial\alpha\partial\beta} + \frac{B_{12}}{R}\frac{\partial u_3}{\partial\alpha} = \boldsymbol{\lambda} u_1,$$

$$-(B_{12}+B_{66})\frac{\partial^2 u_1}{\partial\alpha\partial\beta} - B_{66}\frac{\partial^2 u_2}{\partial\alpha^2} - B_{22}\frac{\partial^2 u_2}{\partial\beta^2} + \frac{B_{22}}{R}\frac{\partial u_3}{\partial\beta} - \frac{\mu^4}{R^2}\left(4B_{66}\frac{\partial^2 u_2}{\partial\alpha^2} + B_{22}\frac{\partial^2 u_2}{\partial\beta^2}\right) - \frac{\mu^4}{R}\left(B_{22}\frac{\partial^3 u_3}{\partial\beta^3} + (B_{12}+4B_{66})\frac{\partial^3 u_3}{\partial\beta\partial\alpha^2}\right) = \boldsymbol{\lambda} u_2\,, \qquad (1.1)$$

$$\mu^4\left(B_{11}\frac{\partial^4 u_3}{\partial\alpha^4} + 2(B_{12}+2B_{66})\frac{\partial^4 u_3}{\partial\alpha^2\partial\beta^2} + B_{22}\frac{\partial^4 u_3}{\partial\beta^4}\right) + \frac{\mu^4}{R}\left(B_{22}\frac{\partial^3 u_2}{\partial\beta^3} + (B_{12}+4B_{66})\frac{\partial^3 u_2}{\partial\beta\partial\alpha^2}\right) - \frac{B_{12}}{R}\frac{\partial u_1}{\partial\alpha} - \frac{B_{22}}{R}\frac{\partial u_2}{\partial\beta} + \frac{B_{22}}{R^2}u_3 = \boldsymbol{\lambda} u_3$$

Here, u_1, u_2 and u_3 are projections of the displacement vector on the directions α and β and on the normal to the median surface of the panel, respectively; R is the radius of the directing circumference of the median surface; $\mu^4 = \frac{h^2}{12}$ (h is the thickness of the panels); $\boldsymbol{\lambda} = \omega^2\rho$, where ω is the angular frequency, ρ is the density of the material; B_{ij} are the elasticity coefficients. The boundary conditions have the form [16]:

$$u_1|_{\alpha=0} = u_2|_{\alpha=0} = u_3|_{\alpha=0} = \left.\frac{\partial u_3}{\partial \beta}\right|_{\alpha=0} = 0; \tag{1.2}$$

$$u_1|_{\alpha=l} = u_2|_{\alpha=l} = u_3|_{\alpha=l} = \left.\frac{\partial u_3}{\partial \beta}\right|_{\alpha=l} = 0; \tag{1.3}$$

$$\begin{cases} u_2|_{\alpha=0} = 0, \left.\frac{\partial u_1}{\partial \alpha} + \frac{B_{12}}{B_{11}}\left(\frac{\partial u_2}{\partial \beta} - \frac{u_3}{R}\right)\right|_{\alpha=0} = 0 \\ u_3|_{\alpha=0} = 0, \left.\frac{\partial^2 u_3}{\partial \alpha^2} + \frac{B_{12}}{B_{11}}\left(\frac{\partial^2 u_3}{\partial \beta^2} - \frac{1}{R}\frac{\partial u_2}{\partial \beta}\right)\right|_{\alpha=0} = 0 \end{cases} \tag{1.4}$$

$$\begin{cases} u_2|_{\alpha=l} = 0, \left.\frac{\partial u_1}{\partial \alpha} + \frac{B_{12}}{B_{11}}\left(\frac{\partial u_2}{\partial \beta} - \frac{u_3}{R}\right)\right|_{\alpha=l} = 0 \\ u_3|_{\alpha=l} = 0, \left.\frac{\partial^2 u_3}{\partial \alpha^2} + \frac{B_{12}}{B_{11}}\left(\frac{\partial^2 u_3}{\partial \beta^2} - \frac{1}{R}\frac{\partial u_2}{\partial \beta}\right)\right|_{\alpha=l} = 0\,; \end{cases} \tag{1.5}$$

$$\begin{cases} \left.\frac{B_{12}}{B_{22}}\frac{\partial u_1}{\partial \alpha} + \frac{\partial u_2}{\partial \beta} - \frac{u_3}{R}\right|_{\beta=0,s} = 0\,, \left.\frac{B_{12}}{B_{22}}\frac{\partial^2 u_3}{\partial \alpha^2} + \frac{\partial^2 u_3}{\partial \beta^2} + \frac{1}{R}\frac{\partial u_2}{\partial \beta}\right|_{\beta=0,s} = 0, \\ \left.\frac{\partial^3 u_3}{\partial \beta^3} + \frac{B_{12}+4B_{66}}{B_{22}}\frac{\partial^3 u_3}{\partial \beta \partial \alpha^2} + \frac{1}{R}\frac{\partial^2 u_2}{\partial \beta^2} + \frac{4B_{66}}{B_{22}}\frac{1}{R}\frac{\partial^2 u_2}{\partial \alpha^2}\right|_{\beta=0,s} = 0, \\ \left.\frac{\partial u_2}{\partial \alpha} + \frac{\partial u_1}{\partial \beta}\right|_{\beta=0,s} = 0\,. \end{cases} \tag{1.6}$$

Relations (1.2), (1.3) are the conditions of rigid-clamped ends at $\alpha = 0, l$, respectively. The relations (1,2), (1.5) are the conditions of rigid-clamped and hinged ends at $\alpha = 0, l$, relations (1.4), (1.5) are the conditions of hinged ends at $\alpha = 0, l$, while conditions (1.6) indicate that the edge generators $\beta = 0, s$ are free.

2. The Derivation and Analysis of the Characteristic Equations.

Let's formally replace the spectral parameter λ by λ_1, λ_2 and λ_3 in the first, second, and third equations of the system (1.1), respectively. In the future, for all three tasks: (1.1)-(1.3), and (1.6); (1.1), (1.2), (1.4) and (1.6); (1.1), (1.4), (1.5), and (1.6) superscripts 1,2,3 in expressions (4)-(6) and (10) will be skipped. The solutions of system (1.1) for the problems are searched in the form

$$(u_1, u_2, u_3) = \{u_m w'_m(\theta_m\alpha), v_m w_m(\theta_m\alpha), w_m(\theta_m\alpha)\}exp(\theta_m \mathcal{X}\beta), m = \overline{1,\infty}\,. \tag{2.1}$$

Here, $w_m(\theta_m\alpha), m = \overline{1,\infty}$, are determined from (4)-(6). u_m, v_m and $\mathcal{X}$ are unknown constants. For the three problems, the conditions (1.2) - (1.5) are obeyed automatically. Let us insert Eq. (2.1) into Eq. (1.1). The obtained equations are multiplied by the vector functions $\{w'_m(\theta_m\alpha), w_m(\theta_m\alpha), w_m(\theta_m\alpha)\}$ in a scalar way and then integrated in the limits from 0 to l. From the first two equations we get

$$(c_m + \varepsilon_m^2 a^2 g_m d_m)u_m = \varepsilon_m \left\{a_m + \frac{B_{22}(B_{12}+B_{66})}{B_{11}B_{66}} a^2 \mathcal{X}^2 l_m + \frac{B_{22}B_{12}}{B_{11}B_{66}} \varepsilon_m^2 a^2 d_m\right\}, \quad (2.2)$$

$$(c_m + \varepsilon_m^2 a^2 g_m d_m)v_m = \varepsilon_m \mathcal{X}\{b_m - a^2 g_m l_m\}. \quad (2.3)$$

From the third equation, by considering the relations (2.2) and (2.3), the characteristic equations are obtained

$$R_{mm}c_m + \varepsilon_m^2 \left\{c_m - b_m\mathcal{X}^2 + \frac{B_{12}}{B_{22}}\beta'_m a_m + a^2[R_{mm}g_m d_m + 2l_m b_m \mathcal{X}^2] + \right.$$

$$\left. \varepsilon_m^2 a^2 d_m \left(b_m - \frac{B_{12}}{B_{11}}\beta'_m\right) - a^4 g_m l_m^2 \mathcal{X}^2\right\} = 0, m = \overline{1,\infty}\,. \quad (2.4)$$

$$a_m = -\left(\frac{B_{22}}{B_{11}}\mathcal{X}^2 - \frac{B_{12}}{B_{11}}(\boldsymbol{\eta}_{2m}^2 - \beta'_m)\right);\; b_m = \frac{B_{22}}{B_{11}}(\mathcal{X}^2 + \boldsymbol{\eta}_{1m}^2) - B_1;$$

$$d_m = \mathcal{X}^2 - \frac{4B_{66}}{B_{22}}\beta'_m;$$

$$c_m = \frac{B_{22}}{B_{11}}\mathcal{X}^4 - B_2\mathcal{X}^2 + \left(\frac{B_{22}}{B_{11}}\boldsymbol{\eta}_{1m}^2 + \frac{B_{66}}{B_{11}}\boldsymbol{\eta}_{2m}^2\right)\mathcal{X}^2 + (\beta'_m - \boldsymbol{\eta}_{2m}^2)\left(\beta''_m - \frac{B_{66}}{B_{11}}\boldsymbol{\eta}_{1m}^2\right);$$

$$B_1 = \frac{B_{11}B_{22}\beta''_m - B_{12}^2\beta'_m - B_{12}B_{66}\beta'_m}{B_{11}B_{66}}, B_2 = \frac{B_{11}B_{22}\beta''_m - B_{12}^2\beta'_m - 2B_{12}B_{66}\beta'_m}{B_{11}B_{66}}; \quad (2.5)$$

$$g_m = \frac{B_{22}}{B_{11}}\mathcal{X}^2 - \frac{B_{22}}{B_{66}}\beta''_m + \frac{B_{22}}{B_{11}}\boldsymbol{\eta}_{1m}^2;\; l_m = \mathcal{X}^2 - \frac{B_{12}+4B_{66}}{B_{22}}\beta'_m\,;\; \boldsymbol{\eta}_{im}^2 = \frac{\lambda_i}{B_{66}\theta_m^2},\ \mathrm{i}=\overline{1,3}\,;$$

$$R_{mm} = a^2\left(\mathcal{X}^4 - \frac{2(B_{12}+2B_{66})}{B_{22}}\beta'_m\mathcal{X}^2 + \frac{B_{11}}{B_{22}}\beta'_m\beta''_m\right) - \frac{B_{66}}{B_{22}}\boldsymbol{\eta}_{3m}^2;$$

$$a^2 = \frac{h^2}{12}\theta_m^2;\; \varepsilon_m = \frac{1}{R\theta_m}.$$

Let $\mathcal{X}_j, j = \overline{1,4}$, be pairwise different roots of Eq.(2.4) with non-positive real parts and $\mathcal{X}_{j+4} = -\mathcal{X}_j, j = \overline{1,4}$. Let $\left(u_1^{(j)}, u_2^{(j)}, u_3^{(j)}\right), j = \overline{1,8}$, be nontrivial solutions of type (2.1) of the system (1.1) at $\mathcal{X} = \mathcal{X}_j, j = \overline{1,8}$, respectively. The solutions of the problems are searched in the form

$$u_i = \sum_{j=1}^{8} u_i^{(j)} w_j\,, i = \overline{1,3}. \quad (2.6)$$

Let us insert Eq. (2.6) into the boundary conditions (1.6). Each of the obtained equation is multiplied by $w_m(\theta_m\alpha)$ or by $w'_m(\theta_m\alpha)$ respectively, and then integrated in the limits from 0 to l. As a result, we obtain the systems of equations

$$\left.\begin{array}{l} \sum_{j=1}^{8} \frac{M_{ij}^{(m)} w_j}{c_m^{(j)} + \varepsilon_m^2 a^2 g_m^{(j)} d_m^{(j)}} = 0, i = \overline{1,4} \\ \sum_{j=1}^{8} \frac{M_{ij}^{(m)} \exp(\theta_m \mathcal{X}_j s) w_j}{c_m^{(j)} + \varepsilon_m^2 a^2 g_m^{(j)} d_m^{(j)}} = 0, i = \overline{5,8} \end{array}\right\} m = \overline{1,\infty}\,, \quad (2.7)$$

$$M_{1j}^{(m)} = \mathcal{X}_j^2 b_m^{(j)} - \frac{B_{12}}{B_{22}} a_m^{(j)}\beta'_m - c_m^{(j)} - a^2 l_m^{(j)} \mathcal{X}_j^2 b_m^{(j)} - \varepsilon_m^2 a^2 d_m^{(j)}\left(b_m^{(j)} - \frac{B_{12}}{B_{22}}\beta'_m\right),$$

$$M_{2j}^{(m)} = \mathcal{X}_j \left\{a_m^{(j)} + b_m^{(j)} + a^2 l_m^{(j)} \left(\frac{B_{12}B_{22}}{B_{11}B_{66}}\mathcal{X}_j^2 + \frac{B_{22}}{B_{66}}\beta_m'' - \frac{B_{22}}{B_{11}}\boldsymbol{\eta}_{1m}^2\right) + \frac{B_{12}B_{22}}{B_{11}B_{66}}\varepsilon_m^2 a^2 d_m^{(j)}\right\},$$
$$M_{3j}^{(m)} = \left(\mathcal{X}_j^2 - \frac{B_{12}}{B_{22}}\beta_m'\right) c_m^{(j)} + \varepsilon_m^2 \left(\mathcal{X}_j^2 b_m^{(j)} + a^2 \frac{4\,B_{12}B_{66}}{(B_{22})^2} g_m^{(j)} (\beta_m')^2\right), \qquad (2.8)$$
$$M_{4j}^{(m)} = \mathcal{X}_j \left(l_m^{(j)} c_m^{(j)} + \varepsilon_m^2 b_m^{(j)} d_m^{(j)}\right),\ M_{4+ij}^{(m)} = M_{ij}^{(m)},\ \text{i}=\overline{1,4}\,.$$
The superscript j in parentheses means that the corresponding function is taken at $\mathcal{X} = \mathcal{X}_j$. For the set of systems (2.7) has a nontrivial solution, it is necessary and sufficient that
$$\Delta = exp\left(-\sum_{j=1}^{4} z_j\right) Det\left\|T_{ij}\right\|_{i,j=1}^{2} = 0, m = \overline{1,\infty}\,, \qquad (2.9)$$
$$T_{11} = \left\|M_{ij}^{(m)}\right\|_{i,j=1}^{4}, T_{12} = \left\|(-1)^{i-1} M_{ij}^{(m)} \exp(z_j)\right\|_{i,j=1}^{4}, z_j = \theta_m \mathcal{X}_j s,$$
$$T_{21} = \left\|M_{ij}^{(m)} \exp(z_j)\right\|_{i=5,j=1}^{8,4}, T_{22} = \left\|(-1)^{i-1} M_{ij}^{(m)}\right\|_{i=5 j=1}^{8,4}. \qquad (2.10)$$
It is shown numerically that the left side of equality (2.9) becomes small when any two roots of Eq. (2.4) become close to each other. This highly complicates calculations and can lead to false solutions. It turns out that from the left side of Eq. (2.9) a multiplier that tends to zero can be separated when the roots approach each other. Let us introduce the following notations:
$$[z_i z_j] = \frac{\theta_m s\left(\exp(z_i) - \exp(z_j)\right)}{(z_i - z_j)}, [z_i z_j z_k] = \frac{\theta_m s\left([z_i z_j] - [z_i z_k]\right)}{(z_j - z_k)},$$
$$[z_1 z_2 z_3 z_4] = \frac{\theta_m s([z_1 z_2 z_3] - [z_1 z_2 z_4])}{(z_3 - z_4)},$$
$$\sigma_1 = \sigma_1(\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \mathcal{X}_4) = \mathcal{X}_1 + \mathcal{X}_2 + \mathcal{X}_3 + \mathcal{X}_4,$$
$$\sigma_2 = \sigma_2(\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \mathcal{X}_4) = \mathcal{X}_1\mathcal{X}_2 + \mathcal{X}_1\mathcal{X}_3 + \mathcal{X}_1\mathcal{X}_4 + \mathcal{X}_2\mathcal{X}_3 + \mathcal{X}_2\mathcal{X}_4 + \mathcal{X}_3\mathcal{X}_4\,,$$
$$\sigma_3 = \sigma_3(\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \mathcal{X}_4) = \mathcal{X}_1\mathcal{X}_2\mathcal{X}_3 + \mathcal{X}_1\mathcal{X}_2\mathcal{X}_4 + \mathcal{X}_1\mathcal{X}_3\mathcal{X}_4 + \mathcal{X}_2\mathcal{X}_3\mathcal{X}_4,$$
$$\sigma_4 = \sigma_4(\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \mathcal{X}_4) = \mathcal{X}_1\mathcal{X}_2\mathcal{X}_3\mathcal{X}_4\,,$$
$$\bar{\sigma}_k = \sigma_k(\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, 0), \bar{\bar{\sigma}}_k = \sigma_k(\mathcal{X}_1, \mathcal{X}_2, 0,0),\ \text{k}=\overline{1,4}\,. \qquad (2.11)$$

In this case, $\bar{\sigma}_4 = \bar{\bar{\sigma}}_4 = \bar{\bar{\sigma}}_3 = 0$. Let $f_n, n = \overline{1,6}$, be a symmetric polynomial of nth order in variables $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \mathcal{X}_4$. It is known that it can be uniquely expressed in terms of elementary symmetric polynomials. By introducing the notations
$$f_n = f_n(\sigma_1, \sigma_2, \sigma_3, \sigma_4);\ \bar{f}_n = f_n(\bar{\sigma}_1, \bar{\sigma}_2, \bar{\sigma}_3, 0), \bar{\bar{f}}_n = f_n(\bar{\bar{\sigma}}_1, \bar{\bar{\sigma}}_2, 0,0), n = \overline{1,6}\,;$$
$$f_1 = \sigma_1;\ f_2 = \sigma_1^2 - \sigma_2\,;\ f_3 = \sigma_1^3 - 2\sigma_1\sigma_2 + \sigma_3;$$
$$f_4 = \sigma_1^4 - 3\sigma_1^2\sigma_2 + \sigma_2^2 + 2\sigma_1\sigma_3 - \sigma_4; \qquad (2.12)$$
$$\bar{f}_5 = \bar{\sigma}_1{}^5 - 4\bar{\sigma}_1{}^3\bar{\sigma}_2 + 3\bar{\sigma}_1\bar{\sigma}_2{}^2 + 3\bar{\sigma}_1{}^2\bar{\sigma}_3 - 2\bar{\sigma}_2\bar{\sigma}_3\,;$$
$$\bar{\bar{f}}_6 = \bar{\bar{\sigma}}_1{}^6 - 5\bar{\bar{\sigma}}_1{}^4\bar{\bar{\sigma}}_2 + 6\bar{\bar{\sigma}}_1{}^2\bar{\bar{\sigma}}_2{}^2 - \bar{\bar{\sigma}}_2{}^3, \qquad (2.13)$$
and performing elementary operations with columns of determinant (2.9), we obtain
$$Det\left\|T_{ij}\right\|_{i,j=1}^{2} = K^2 Det\left\|m_{ij}\right\|_{i,j=1}^{8}, m = \overline{1,\infty}\,, \qquad (2.14)$$
$$K = (\mathcal{X}_1 - \mathcal{X}_2)(\mathcal{X}_1 - \mathcal{X}_3)(\mathcal{X}_1 - \mathcal{X}_4)(\mathcal{X}_2 - \mathcal{X}_3)(\mathcal{X}_2 - \mathcal{X}_4)(\mathcal{X}_3 - \mathcal{X}_4). \qquad (2.15)$$
Expressions for m_{ij} are given in the Appendix. Equations (2.9) are equivalent to equations
$$Det\left\|m_{ij}\right\|_{i,j=1}^{8} = 0, m = \overline{1,\infty}\,. \qquad (2.16)$$

By considering the possible relations between $\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2$ and $\boldsymbol{\lambda}_3$ we conclude that equations (2.16) determine frequencies of the corresponding types of vibrations. For $\boldsymbol{\lambda}_1 = \boldsymbol{\lambda}_2 = \boldsymbol{\lambda}_3 = \boldsymbol{\lambda}$, equations (2.4) become the characteristic equations of system (1.1), and equations (2.16) become the dispersion equations of problems (1.1)- (1.6), respectively.

In Section 5, the asymptotes of the dispersion equations (2.16) for $\varepsilon_m = \frac{1}{\theta_m R} \to 0$ (transition to a rectangular plates with arbitrary fastening of the opposite sides or to vibrations localized at the free edges of the cylindrical panels with arbitrary fastening of the ends) and for $\theta_m s \to \infty$ (transition to a wide enough cylindrical panels with arbitrary fastening of the ends or to vibrations localized at the free edges of the cylindrical panels with arbitrary fastening of the ends) are investigated. For checking the reliability of the asymptotic relations found in Section 5, the free planar and bending vibrations of rectangular plates with arbitrary fastening of the opposite sides are investigated in the next two sections.

3. Planar vibrations of an orthotropic rectangular plate with arbitrary fastening of the opposite sides.

Let an orthotropic rectangular plate is defined in a three-orthogonal system of rectilinear coordinates (α, β, γ) with the origin on the free face plane such that the coordinate plane $\alpha\beta$ coincides with the median plane of the plate and the principal axes of symmetry of the material are aligned with the coordinate lines (Fig. 2). Let s_0 and l be the width and the length of the plate, respectively. The problems of existence of free planar vibrations of rectangular plates with arbitrary fastening of the opposite sides are investigated. As the initial equations consider the equations of low-amplitude planar vibrations of the classical theory of orthotropic plates [16]

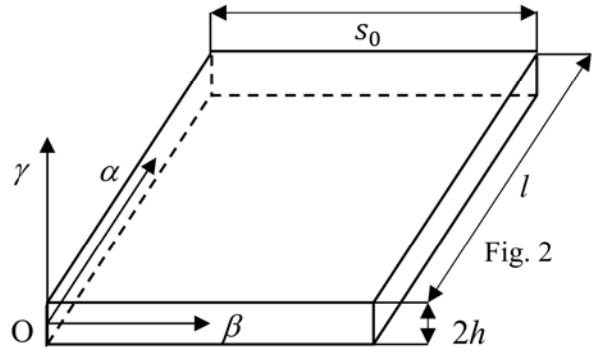

Fig. 2

$$-B_{11}\frac{\partial^2 u_1}{\partial \alpha^2} - B_{66}\frac{\partial^2 u_1}{\partial \beta^2} - (B_{12} + B_{66})\frac{\partial^2 u_2}{\partial \alpha \partial \beta} = \boldsymbol{\lambda} u_1,$$

$$-(B_{12} + B_{66})\frac{\partial^2 u_1}{\partial \alpha \partial \beta} - B_{66}\frac{\partial^2 u_2}{\partial \alpha^2} - B_{22}\frac{\partial^2 u_2}{\partial \beta^2} = \boldsymbol{\lambda} u_2 \,. \tag{3.1}$$

Here $\alpha(0 \le \alpha \le l)$ and $\beta(0 \le \beta \le s_0)$ are the orthogonal rectilinear coordinates of a point on the middle plane; u_1 and u_2 are the displacements in α and β directions, respectively; $B_{ik}, k = 1,2,6$ are the coefficients of elasticity; $\boldsymbol{\lambda} = \boldsymbol{\omega}^2 \boldsymbol{\rho}$, where $\boldsymbol{\omega}$ is the natural frequency; $\boldsymbol{\rho}$ is the density of material. The boundary conditions have the form [16]

$$u_1|_{\alpha=0} = u_2|_{\alpha=0} = 0, \tag{3.2}$$

$$u_1|_{\alpha=l} = u_2|_{\alpha=l} = 0, \tag{3.3}$$

$$u_2|_{\alpha=0} = 0, \frac{\partial u_1}{\partial \alpha} + \frac{B_{12}}{B_{11}}\frac{\partial u_2}{\partial \beta}\Big|_{\alpha=0} = 0, \tag{3.4}$$

$$u_2|_{\alpha=l} = 0, \frac{\partial u_1}{\partial \alpha} + \frac{B_{12}}{B_{11}}\frac{\partial u_2}{\partial \beta}\Big|_{\alpha=l} = 0\,, \tag{3.5}$$

$$\frac{B_{12}}{B_{22}}\frac{\partial u_1}{\partial \alpha} + \frac{\partial u_2}{\partial \beta}\Big|_{\beta=0,s_0} = 0\,,\ \frac{\partial u_2}{\partial \alpha} + \frac{\partial u_1}{\partial \beta}\Big|_{\beta=0,s_0} = 0\,. \tag{3.6}$$

Relations (3.2), (3.3) are the conditions of rigid-clamped sides at $\alpha = 0, l$, respectively. Relations (3.2), (3.5) are the conditions of rigid-clamped and hinged sides at $\alpha = 0, l$. Relations (3.4), (3.5) are the conditions of hinged sides at $\alpha = 0, l$, while conditions (3.6) indicate that the sides $\beta = 0, s_0$ are free.

The problems (3.1)-(3.3), (3.6); (3.1), (3.2), (3.5), (3.6) does not allow separation of variables. The differential operators corresponding to these problems and the problem (3.1), (3.4)-(3.6) are self-conjugate and nonnegative definite. Therefore, the generalized Kantorovich-Vlasov method of the reduction to ordinary differential equations can be used to find vibration eigenfrequencies and eigenmodes [11-15].

In the future, for all three problems, superscripts 1, 2, 3 in expressions (4)-(6), (10) will be skipped. The solutions of system (3.1) for problems are searched in the form

$$(u_1, u_2) = \{u_m w_m'(\theta_m\alpha), v_m w_m(\theta_m\alpha)\}exp(\theta_m y\beta), m = \overline{1,\infty}\,. \tag{3.7}$$

Here, $w_m(\theta_m\alpha), m = \overline{1,\infty}$, are determined from (4)-(6). u_m, v_m and y are unknown constants. For the three problems, the boundary conditions (3.2)-(3.5) are satisfied automatically. Let us insert (3.7) into Eq. (3.1). Then, the obtained equations are multiplied by vector function $\{w_m'(\theta_m\alpha), w_m(\theta_m\alpha)\}$ in a scalar way and integrated in the limits from 0 to l. As a result, the following systems of equations are obtained

$$\left(\beta_m'' - \frac{B_{66}}{B_{11}}(y^2 + \boldsymbol{\eta}_m^2)\right)u_m - \frac{B_{12}+B_{66}}{B_{11}}yv_m = 0,$$

$$\frac{B_{12}+B_{66}}{B_{22}}\beta_m' y u_m - \left(y^2 - \frac{B_{66}}{B_{22}}(\beta_m' - \boldsymbol{\eta}_m^2)\right)v_m = 0, m = \overline{1,\infty}\,. \tag{3.8}$$

where $\boldsymbol{\eta}_m^2 = \frac{\lambda}{(\theta_m^2 B_{66})}$. θ_m and β_m', β_m'' are determined in equations (7)-(10), respectively.

By equating the determinant of system (3.8) to zero, the following characteristic equations of the system of equations (3.1) are found:

$$c_m = \frac{B_{22}}{B_{11}}y^4 - B_2 y^2 + \frac{B_{22}+B_{66}}{B_{11}}\boldsymbol{\eta}_m^2 y^2 + (\beta_m' - \boldsymbol{\eta}_m^2)\left(\beta_m'' - \frac{B_{66}}{B_{11}}\boldsymbol{\eta}_m^2\right) = 0,$$

$$m = \overline{1,\infty}\,. \tag{3.9}$$

Let y_1 and y_2 be various roots of Eq. (3.9) with non-positive real parts and $y_{2+j} = -y_j$, j=1,2. As the solutions of systems (3.8) for $= y_j, j = \overline{1,4}$, we take

$$u_m^{(j)} = y_j^2 - \frac{B_{66}}{B_{22}}(\beta_m' - \boldsymbol{\eta}_m^2), v_m^{(j)} = \frac{B_{12}+B_{66}}{B_{22}}\beta_m' y_j,\ j = \overline{1,4}; m = \overline{1,\infty}. \tag{3.10}$$

The solutions of the problems (3.1) -(3.6) can be presented in the form

$$(u_1, u_2) =$$

$$\left\{\sum_{j=1}^4 u_m^{(j)} w_m'(\theta_m\alpha)\exp(\theta_m y_j\beta)\, w_j, \sum_{j=1}^4 v_m^{(j)} w(\theta_m\alpha)\exp(\theta_m y_j\beta)\, w_j,\right\} \tag{3.11}$$

Let us insert (3.11) into the boundary conditions (3.6). Each of the obtained equation is multiplied by $w(\theta_m\alpha)$ or by $w_m'(\theta_m\alpha)$, and then integrated in the limits from 0 to l. As a result, the following systems of equations are obtained

$$\begin{cases} \sum_{j=1}^{4} R_{1j}^{(m)} w_j = 0, \sum_{j=1}^{4} R_{1j}^{(m)} \exp(z_j) w_j = 0, \\ \sum_{j=1}^{4} R_{2j}^{(m)} w_j = 0, \sum_{j=1}^{4} R_{2j}^{(m)} \exp(z_j) w_j = 0, \end{cases} m = \overline{1, \infty}. \tag{3.12}$$

$$R_{1j}^{(m)} = y_j^2 + \frac{B_{12}}{B_{22}}(\beta_m' - \boldsymbol{\eta}_m^2), R_{2j}^{(m)} = y_j\left(y_j^2 + \frac{B_{12}}{B_{22}}\beta_m' + \frac{B_{66}}{B_{22}}\boldsymbol{\eta}_m^2\right), z_j = \theta_m y_j s_0, j = \overline{1,4}. \tag{3.13}$$

By equating the determinant $\Delta_e^{(m)}$of the system (3.12) to zero and performing elementary operations with columns of the determinant the following dispersion equations are obtained

$$\Delta_e^{(m)} = exp(-z_1 - z_2)(y_2 - y_1)^2 Det\|l_{ij}\|_{i,j=1}^{4} = 0, m = \overline{1, \infty}. \tag{3.14}$$

$$\begin{gathered} l_{11} = R_{11}^{(m)}, l_{12} = y_1 + y_2, l_{13} = l_{11}\exp(z_1), l_{14} = l_{12}\exp(z_2) + l_{11}[z_1 z_2]; \\ l_{21} = R_{21}^{(m)}, l_{22} = y_1 y_2 + \frac{(B_{11}B_{22}\beta_m'' - B_{12}^2\beta_m' - B_{12}B_{66}\beta_m')}{(B_{22}B_{66})} - \boldsymbol{\eta}_m^2, \\ l_{23} = -l_{21}\exp(z_1), l_{24} = -l_{22}\exp(z_2) - l_{21}[z_1 z_2];\ l_{31} = l_{13}, l_{32} = l_{14}, \\ l_{33} = l_{11}, l_{34} = l_{12}, l_{41} = -l_{23}, l_{42} = -l_{24}, l_{43} = -l_{21}, l_{44} = -l_{22}, \\ [z_1 z_2] = \frac{\theta_m s_0(\exp(z_2) - \exp(z_1))}{(z_2 - z_1)}. \end{gathered} \tag{3.15}$$

The equations (3.14) are equivalent to the equations

$$\begin{gathered} Det\|l_{ij}\|_{i,j=1}^{4} = -\left(\frac{(B_{12}+B_{66})}{B_{22}}\right)^2 K_{2m}^2(\boldsymbol{\eta}_m^2)\left(1 + exp\left(2(z_1 + z_2)\right)\right) - \\ 8l_{21}l_{11}l_{12}l_{22}exp(z_1 + z_2) + (l_{11}l_{22} + l_{12}l_{21})^2(\exp(2z_1) + \exp(2z_2)) + \\ 4l_{11}l_{21}(l_{11}l_{22} + l_{12}l_{21})(\exp(z_2) - \exp(z_1))[z_1 z_2] + 4l_{11}^2 l_{21}^2 [z_1 z_2]^2 = 0, m = \overline{1, \infty}. \end{gathered} \tag{3.16}$$

$$K_{2m}^2(\boldsymbol{\eta}_m^2) = (\beta_m' - \boldsymbol{\eta}_m^2)\left(\frac{B_{11}B_{22}\beta_m'' - B_{12}^2\beta_m'}{B_{22}B_{66}} - \boldsymbol{\eta}_m^2\right) - \boldsymbol{\eta}_m^2 y_1 y_2. \tag{3.17}$$

If y_1 and y_2 are the roots of Eq. (3.9) with negative real parts, then, for $\theta_m s_0 \to \infty$, the roots of Eq. (3.16) are approximated by the roots of the equations

$$K_{2m}^2(\boldsymbol{\eta}_m^2) = (\beta_m' - \boldsymbol{\eta}_m^2)\left(\frac{B_{11}B_{22}\beta_m'' - B_{12}^2\beta_m'}{B_{22}B_{66}} - \boldsymbol{\eta}_m^2\right) - \boldsymbol{\eta}_m^2 y_1 y_2 = 0, m = \overline{1, \infty}. \tag{3.18}$$

The equations (3.18) are the analogue of the Rayleigh equations for a long enough orthotropic rectangular plate with arbitrary fastening of the opposite sides. Thus, the eigenfrequencies of the problems (3.1)-(3.6) can be found from (3.16).

4. Bending vibrations of an orthotropic rectangular plate with arbitrary fastening of the opposite sides.

Consider an orthotropic rectangular plate with thickness h, width s_0, and length l (Fig. 2).

Consider now the problems of the existence of free bending vibrations of a rectangular plate with arbitrary fastening of the opposite sides. Let us start with the equation of low amplitude bending vibrations of the classical theory of orthotropic plates [16]

$$\mu^4\left(B_{11}\frac{\partial^4 u_3}{\partial\alpha^4} + 2(B_{12} + 2B_{66})\frac{\partial^4 u_3}{\partial\alpha^2\partial\beta^2} + B_{22}\frac{\partial^4 u_3}{\partial\beta^4}\right) = \boldsymbol{\lambda} u_3, \tag{4.1}$$

where $\alpha(0 \le \alpha \le l)$ and $\beta(0 \le \beta \le s_0)$ are the orthogonal rectilinear coordinates of a point of the median plane of the plate; u_3 is the normal component of the displacement vector of a point of the median plane; $B_{ik}, k = 1,2,6$ are the elasticity coefficients; $\mu^4 = \frac{h^2}{12}$; $\boldsymbol{\lambda} = \omega^2\rho$, where ω is the natural frequency; ρ is the density of material.

The boundary conditions are given as follows:

$$u_3|_{\alpha=0} = \left.\frac{\partial u_3}{\partial\beta}\right|_{\alpha=0} = 0; \tag{4.2}$$

$$u_3|_{\alpha=l} = \left.\frac{\partial u_3}{\partial \beta}\right|_{\alpha=l} = 0; \tag{4.3}$$

$$u_3|_{\alpha=0} = 0, \left.\frac{\partial^2 u_3}{\partial \alpha^2} + \frac{B_{12}}{B_{11}}\frac{\partial^2 u_3}{\partial \beta^2}\right|_{\alpha=0} = 0\,; \tag{4.4}$$

$$u_3|_{\alpha=l} = 0, \left.\frac{\partial^2 u_3}{\partial \alpha^2} + \frac{B_{12}}{B_{11}}\frac{\partial^2 u_3}{\partial \beta^2}\right|_{\alpha=l} = 0\,; \tag{4.5}$$

$$\left.\frac{B_{12}}{B_{22}}\frac{\partial^2 u_3}{\partial \alpha^2} + \frac{\partial^2 u_3}{\partial \beta^2}\right|_{\beta=0,s_0} = 0, \left.\frac{\partial^3 u_3}{\partial \beta^3} + \frac{B_{12}+4B_{66}}{B_{22}}\frac{\partial^3 u_3}{\partial \beta \partial \alpha^2}\right|_{\beta=0,s_0} = 0. \tag{4.6}$$

Relations (4.2), (4.3) are the conditions of rigid-clamped sides at $\alpha = 0, l$, respectively. Relations (4.2), (4.5) are the conditions of rigid-clamped and hinged sides at $\alpha = 0, l$. Relations (4.4), (4.5) are the conditions of hinged sides at $\alpha = 0, l$, while conditions (4.6) indicate that the sides $\beta = 0, s_0$ are free.

The problems (4.1)-(4.3), (4.6); (4.1), (4.2), (4.5), (4.6) does not allow separation of variables. The differential operators corresponding to these problems and the problem (4.1), (4.4)-(4.6) are self-conjugate and nonnegative definite. Therefore, the generalized Kantorovich-Vlasov method of the reduction to ordinary differential equations can be used to find the vibration eigenfrequencies and eigenmodes [11-15].

In the future, for all three problems, superscripts 1, 2, 3 in expressions (4)-(6), (10) will be skipped.

The solution of the equation (4.1) is searched in the form

$$u_3 = w_m(\theta_m \alpha) exp(\theta_m y \beta), m = \overline{1,\infty}\,, \tag{4.7}$$

where $w_m(\theta_m \alpha)$ are defined in (4)-(6) and y are unknown constant. For three problems, the boundary conditions (4.2)-(4.5) are satisfied automatically. Substitute (4.7) into Eq. (4.1). After multiplying the resulting equation by $w_m(\theta_m \alpha)$ and integrating it in the limits from 0 to l the characteristic equations are obtained

$$R_{mm} = a^2\left(y^4 - \frac{2(B_{12}+2B_{66})}{B_{22}}\beta'_m y^2 + \frac{B_{11}}{B_{22}}\beta'_m\beta''_m\right) - \frac{B_{66}}{B_{22}}\boldsymbol{\eta}_m^2 = 0, m = \overline{1,\infty}\,. \tag{4.8}$$

$$a^2 = \frac{h^2}{12}\theta_m^2\,,\ \boldsymbol{\eta}_m^2 = \frac{\lambda}{B_{66}\theta_m^2}\,, \tag{4.9}$$

where θ_m and $\beta'_m\,, \beta''_m$ are defined in Eq. (7)-(10), respectively. Let y_3 and y_4 be various roots of Eq. (4.8) with non-positive real parts, $y_{2+j} = -y_j, j = 3,4$. Solutions of the problems (4.1)-(4.6) are searched in the form

$$u_3 = \textstyle\sum_{j=3}^{6} w_m(\theta_m \alpha) exp\left(\theta_m y_j \beta'_m\right) w_j\,, m = \overline{1,\infty}. \tag{4.10}$$

By inserting Eq. (4.10) into the boundary conditions (4.6) and after multiplying the resulting equations by $w_m(\theta_m \alpha)$, and integrating them in the limits from 0 to l, the systems of equations are obtained

$$\begin{cases} \sum_{j=3}^{6} \boldsymbol{R}_{3j}^{(m)} w_j = 0, \sum_{j=3}^{6} \boldsymbol{R}_{4j}^{(m)} w_j = 0, \\ \sum_{j=3}^{6} \boldsymbol{R}_{3j}^{(m)} exp\left(z_j\right) w_j = 0, \sum_{j=3}^{6} \boldsymbol{R}_{4j}^{(m)} exp\left(z_j\right) w_j = 0, \end{cases} m = \overline{1,\infty}. \tag{4.11}$$

$$\boldsymbol{R}_{3j}^{(m)} = y_j^2 - \frac{B_{12}}{B_{22}}\beta'_m\,, \boldsymbol{R}_{4j}^{(m)} = y_j^3 - \frac{B_{12}+4B_{66}}{B_{22}}\beta'_m y_j\,, z_j = \theta_m y_j s_0\,, j = \overline{3,6}. \tag{4.12}$$

By equating the determinants $\Delta_b^{(m)}$ of systems (4.9) to zero and performing elementary operations on the columns of the determinant, the dispersion equations are obtained

$$\Delta_b^{(m)} = exp(-z_3 - z_4)(y_4 - y_3)^2 Det\left\|b_{ij}\right\|_{i,j=1}^{4} = 0, m = \overline{1,\infty}. \tag{4.13}$$

$$b_{11} = R_{33}^{(m)}, b_{12} = y_3 + y_4, b_{13} = b_{11}\exp(z_3),$$
$$b_{14} = b_{12}\exp(z_4) + b_{11}[z_3 z_4];\ b_{21} = R_{43}^{(m)}, b_{22} = y_3 y_4 + \beta'_m \frac{B_{12}}{B_{22}},$$
$$b_{23} = -b_{21}\exp(z_3), b_{24} = -b_{22}\exp(z_4) - l_{21}[z_3 z_4];$$
$$b_{31} = b_{13}, b_{32} = b_{14}, b_{33} = b_{11}, b_{34} = b_{12}, b_{41} = -b_{23}, b_{42} = -b_{24}, \quad (4.14)$$
$$b_{43} = -b_{21}, b_{44} = -b_{22}, [z_3 z_4] = \frac{\theta_m s_0(\exp(z_4) - \exp(z_3))}{(z_4 - z_3)}.$$

The equations (4.13) are equivalent to the equations

$$Det\|b_{ij}\|_{i,j=1}^{4} = -K_{1m}^2(\boldsymbol{\eta}_m^2)\left(1 + exp\left(2(z_3 + z_4)\right)\right) -$$
$$8b_{11}b_{12}b_{21}b_{22}exp(z_1 + z_2) + (b_{11}b_{22} + b_{12}b_{21})^2 \times \quad (4.15)$$
$$(\exp(2z_3) + \exp(2z_4)) + 4b_{11}b_{21}(b_{11}b_{22} + b_{12}b_{21})(\exp(z_4) - \exp(z_3))[z_3 z_4] +$$
$$4b_{11}^2 b_{21}^2 [z_3 z_4]^2 = 0, m = \overline{1, \infty}.$$

$$K_{1m}(\boldsymbol{\eta}_m^2) = y_3^2 y_4^2 + 4\frac{B_{66}}{B_{22}}\beta'_m\, y_3 y_4 - \left(\frac{B_{12}}{B_{22}}\right)^2 (\beta'_m)^2. \quad (4.16)$$

If y_3 and y_4 are the roots of Eq. (4.8) with negative real parts, then, at $\theta_m s_0 \to \infty$, the roots of Eq. (4.15) are approximated by the roots of the equations

$$K_{1m}(\boldsymbol{\eta}_m^2) = y_3^2 y_4^2 + 4\frac{B_{66}}{B_{22}}\beta'_m\, y_3 y_4 - \left(\frac{B_{12}}{B_{22}}\right)^2 (\beta'_m)^2 = 0, m = \overline{1, \infty}. \quad (4.17)$$

The equations (4.17) are the analogues of the Konenkov equations for a long enough orthotropic rectangular plate with arbitrary fastening of the opposite sides (compare with [7-10, 19-20]). Thus, eigenfrequencies of the problems (4.1)-(4.6) can be found from (4.15).

5. Asymptotics of the dispersion equations (2.16) in the limit $\varepsilon_m \to 0$.

Using the previous formulas, we assume that $\eta_{1m} = \eta_{2m} = \eta_{3m} = \eta_m$. Then, as $\varepsilon_m \to 0$ Eqs. (2.4) transform into

$$c_m = \frac{B_{22}}{B_{11}}y^4 - B_2 y^2 + \frac{B_{22} + B_{66}}{B_{11}}\boldsymbol{\eta}_m^2 y^2 +$$
$$(\beta'_m - \boldsymbol{\eta}_m^2)\left(\beta''_m - \frac{B_{66}}{B_{11}}\boldsymbol{\eta}_m^2\right) = 0, m = \overline{1, \infty}; \quad (5.1)$$
$$R_{mm} = a^2\left(y^4 - \frac{2(B_{12} + 2B_{66})}{B_{22}}\beta'_m y^2 + \frac{B_{11}}{B_{22}}\beta'_m\beta''_m\right)$$
$$-\frac{B_{66}}{B_{22}}\boldsymbol{\eta}_m^2 = 0, m = \overline{1, \infty}. \quad (5.2)$$

Here the limiting process $\varepsilon_m \to 0$ is understood in the sense that by fixing the radius R and $b = s_0$ – the distance between the boundary generatrixes of the cylindrical panel, a transition to a cylindrical panel of radius $R' = nR$ and to the limit $\varepsilon'_m = \frac{1}{n\theta_m R} = \frac{\varepsilon_m}{n} \to 0$ for $n \to \infty$ is performed.

The equations (5.1) and (5.2) are the characteristic equations of the equations of planar and bending vibrations of orthotropic plates with arbitrary fastening of the opposite sides, respectively. The roots of Eqs. (5.1) and (5.2) with non-positive real parts, as in Sections 3 and 4, are denoted by y_1, y_2 and y_3, y_4, respectively. In the same way as in [18], it is proved that for

$$\varepsilon_m \ll 1;\ y_i \neq y_j, i \neq j, \quad (5.3)$$

the roots $\mathcal{X}^2$ of Eqs. (2.4) can be presented as

$$\mathcal{X}_i^2 = y_i^2 + \alpha_i^{(m)}\varepsilon_m^2 + \beta_i^{(m)}\varepsilon_m^4 + \cdots, i = \overline{1,4}, m = \overline{1,\infty}. \quad (5.4)$$

Under the condition (5.3), considering the relations (2.8), (2.14) and (5.4) and the fact that

$$M_{3j}^{(m)} = M_{4j}^{(m)} = O(\varepsilon_m^2), j = 1,2 \quad (5.5)$$

Eq. (2.16) can be reduced to the form

$$Det\|m_{ij}\|_{i,j=1}^{8} = N^2(\boldsymbol{\eta}_m^2)K_{3m}^2(\boldsymbol{\eta}_m^2)Det\|l_{ij}\|_{i,j=1}^{4}Det\|b_{ij}\|_{i,j=1}^{4} + O(\varepsilon_m^2) = 0, m = \overline{1,\infty} \quad (5.6)$$

where $Det\|l_{ij}\|_{i,j=1}^{4}$ and $Det\|b_{ij}\|_{i,j=1}^{4}$ are determined by (3.16) and (4.15), respectively, and

$$N(\boldsymbol{\eta}_m^2) = (y_3+y_1)(y_3+y_2)(y_4+y_1)(y_4+y_2),$$

$$K_{3m}(\boldsymbol{\eta}_m^2) = \Bigg\{(\beta_m' - \boldsymbol{\eta}_m^2)\left(\frac{B_{11}}{B_{22}}\beta_m'' - \frac{B_{66}}{B_{22}}\boldsymbol{\eta}_m^2\right)\left(\frac{B_{22}}{B_{11}} + a^2\left(\frac{B_{66}}{B_{11}}\boldsymbol{\eta}_m^2 + \frac{B_{12}^2+3B_{12}B_{66}+4B_{66}^2}{B_{11}B_{66}}\beta_m'\right)\right)^2 + \left(\frac{B_{11}B_{11}\beta_m''-B_{12}^2\beta_m'-2B_{12}B_{66}\beta_m'}{B_{22}B_{66}} - \frac{B_{22}+B_{66}}{B_{22}}\boldsymbol{\eta}_m^2\right) \times$$

$$\left(\frac{B_{22}}{B_{11}} + a^2\left(\frac{B_{66}}{B_{11}}\boldsymbol{\eta}_m^2 + \frac{B_{12}^2+3B_{12}B_{66}+4B_{66}^2}{B_{11}B_{66}}\beta_m'\right)\right) \times$$

$$\left(\frac{B_{22}}{B_{11}}\boldsymbol{\eta}_m^2 - B_1 + a^2\left(\frac{B_{66}}{B_{11}}\boldsymbol{\eta}_m^2 - \beta_m''\right)\left(\boldsymbol{\eta}_m^2 + \frac{B_{12}+3B_{66}}{B_{66}}\beta_m'\right)\right) + \quad (5.7)$$

$$\left(\frac{B_{22}}{B_{11}}\boldsymbol{\eta}_m^2 - B_1 + a^2\left(\frac{B_{66}}{B_{11}}\boldsymbol{\eta}_m^2 - \beta_m''\right)\left(\boldsymbol{\eta}_m^2 + \frac{B_{12}+3B_{66}}{B_{66}}\beta_m'\right)\right)^2\Bigg\}\left(\frac{B_{22}}{(B_{12}+B_{66})\beta_m'}\right)^2.$$

From Eq. (5.6), it follows that in the limit $\boldsymbol{\varepsilon_m} \to \mathbf{0}$, Eqs. (2.16) decompose into the totality of equations

$$Det\|l_{ij}\|_{i,j=1}^{4} = 0, m = \overline{1,\infty}; Det\|b_{ij}\|_{i,j=1}^{4} = 0, m = \overline{1,\infty};$$
$$K_{3m}(\boldsymbol{\eta}_m^2) = 0, m = \overline{1,\infty}. \quad (5.8)$$

Here, the first two equations are the dispersion equations of the planar and bending vibrations, respectively, as in the similar problems of an orthotropic rectangular plate with arbitrary fastening of the opposite sides.

The roots of the third equation correspond to planar vibrations of a cylindrical panel. The third equation appears as the result of using the equation of the corresponding classical theory of orthotropic cylindrical shells.

If y_1, y_2 and y_3, y_4 are the roots of the Eqs. (5.1) and (5.2), respectively, with negative real parts, then, for $\theta_m s \to \infty$, Eqs. (2.16) and (5.6) will be transformed into the equations

$$Det\|m_{ij}\|_{i,j=1}^{8} = \left(\frac{(B_{12}+B_{66})}{B_{22}}\right)^2 N^2(\boldsymbol{\eta}_m^2)K_{1m}^2(\boldsymbol{\eta}_m^2)K_{2m}^2(\boldsymbol{\eta}_m^2)K_{3m}^2(\boldsymbol{\eta}_m^2) + O(\varepsilon_m^2) + \sum_{j=1}^{4} O\big(\exp(z_j)\big) = 0, \ m = \overline{1,\infty}. \quad (5.9)$$

From Eqs. (5.9), it follows that for $\varepsilon_m \to 0$ and $\theta_m s \to \infty$ the roots of dispersion equations (2.16) are approximated by the roots of the equations

$$K_{1m}(\boldsymbol{\eta}_m^2) = 0, = \overline{1,\infty}, K_{2m}(\boldsymbol{\eta}_m^2) = 0, = \overline{1,\infty}, K_{3m}(\boldsymbol{\eta}_m^2) = 0, = \overline{1,\infty}. \quad (5.10)$$

The first two equations of (5.10) are the dispersion equations of bending and planar vibrations of wide enough orthotropic rectangular plate with arbitrary fastening of the opposite sides (see Eqs. (4.15) and (3.16)). Hence, for small ε_m and large $\theta_m s$, the approximate values of roots of Eqs. (2.16) correspond to the roots of Eqs. (5.8) and (5.10).

6. Numerical Results.

In the Table 1 some dimensionless characteristics of eigenvalues $\boldsymbol{\eta}_m$ for predominantly bending, predominantly planar and nonsymmetrical vibrations of an orthotropic cylindrical boron plastic panel for problem (1.1)-(1.3), (1.6), are given with mechanical properties:

$$\rho = 2.10^3 \frac{kg}{M^3};\ E_1 = 2.646.10^{11} \frac{N}{M^2};\ E_2 = 1.323.\frac{10^{10}N}{M^2}$$

$$G = 9.604.\frac{10^9 N}{M^2};\ \sigma_1 = 0.2;\ \sigma_2 = 0.012; \quad (6.1)$$

and geometrical parameters: $R = 40, l = 4, s = 5.00326, s = 15.0893.$

In Tables 2 and 3 some dimensionless characteristics of the eigenvalues $\boldsymbol{\eta}_m$ for predominantly bending, predominantly planar and nonsymmetrical vibrations of the orthotropic cylindrical boron plastic panels for problems (1.1), (1.2), (1.4), (1.6); (1.1), (1.4), (1.5), (1.6) with the same mechanical and geometrical parameters are given.

Comment: the eigenvalues $\boldsymbol{\eta}_m$ which corresponds to roots of equations $K_{3m}(\boldsymbol{\eta}_m^2) = 0, m = \overline{1, \infty}$ have other values of the order 10^8, which are not given.

Table 1. Characteristics of eigenfrequencies for predominantly bending, predominantly planar and nonsymmetrical vibrations of a cylindrical boron plastic panel with rigid-clamped ends, when $\boldsymbol{l = 4, s = 5.00326, s = 15.0893}$.

m	θ_m^1 $\beta_m'^1$ $\beta_m''^1$	$\boldsymbol{\eta_{1m} = \eta_{2m} = 0}$, $\boldsymbol{\eta_{3m} = \eta_m}$. $\boldsymbol{s = 5.00326}$; **s=15.0893.**	$\boldsymbol{\eta_{1m} = \eta_{2m} = \eta_m}$, $\boldsymbol{\eta_{3m} = 0}$. $\boldsymbol{s = 5.00326}$; **s=15.0893.**	$\boldsymbol{\eta_{1m} = \eta_{2m} = \eta_{3m} = \eta_m}$. $\boldsymbol{s = 5.00326}$; **s=15.0893.**
1	1.18251 0.96565 0.96499	0.03727 b, 0.03767 b; 0.04437 b, 0.04670 b	4.99154 n	0.03716 b, 0.03754b; 0.04433 b, 0.04667b, 4.99154 n
2	1.96330 0.74537 1.34161	0.06145 b, 0.06175 b; 0.06701 b, 0.06869 b	5.97373 n	0.06140 b, 0.06199 b; 0.06723 b, 0.06889 b, 5.97373 n
3	2.74890 0.99993 0.99993	0.08411 b, 0.08521 b; 0.08431 b, 0.08456 b	5.08194 n	0.08410 b, 0.08521 b; 0.08431 b, 0.08456 b, 5.08194 n
4	3.53429 0.61849 1.61684	0.10921 b, 0.11051 b; 0.10794 b, 0.10815 b	6.59674 n	0.10809 b, 0.10904 b; 0.10761 b, 0.10770 b, 6.59674 n
5	4.31969 1. 1.	0.13187 b, 0.13206 b; 0.13159 b, 0.13182 b	-, -; 0.98692 e, 0.98693 e 5.08340 n	0.13118 b, 0.13137 b, -, -; 0.13142 b, 0.13159 b, 0.98692 e, 0.98692 e 5.08340 n
16	12.9591 1. 1.	0.40715 b, 0.40962 b; 0.40054 b, 0.40109 b	0.98735 e, 0.98735 e;	0.40715 b, 0.40962 b, 0.98735 e, 0.98735 e; 0.40053 b, 0.40108 b,

			0.98696 e, 0.98696 e 5.09503 n	0.98735 e, 0.98735 e 5.09503 n
17	13.7445 1. 1.	0.41757 b, 0.41808 b; 0.41789 b, 0.41808 b	0.98695 e, 0.98696 e; 0.98696 e, 0.98696 e 5.10100 n	0.41757 b, 0.41808 b, 0.98696 e, 0.98696 e; 0.41790 b, 0.41808 b, 0.98696 e, 0.98696 e 5.10304 n
18	14.5299 1. 1.	0.44136 b, 0.44184 b; 0.44076 b, 0.44151b	0.98695 e, 0.98696 e; 0.998696 e, 0.98696 e 5.10304 n	0.44136 b, 0.44184 b, 0.98696 e, 0.98696 e; 0.44083 b, 0.44151 b, 0.98696 e, 0.98696 e 5.10304 n
19	15.3153 1. 1.	0.46458 b, 0.46561 b; 0.46459 b, 0.46462 b	0.98696 e, 0.98696 e; 0.98696 e, 0.98696 e 5.10515 n	0.46458 b, 0.46561 b, 0.98696 e, 0.98696 e; 0.46459 b, 0.46462 b, 0.98696 e, 0.98696 e 5.10515 n
20	15.1007 1. 1.	0.48938 b, 0.48994 b; 0.48841 b, 0.48842 b	0.98696 e, 0.98696 e; 0.98696 e, 0.98696 e 5.10730 n	0.48938 b, 0.48994 b, 0.98696 e, 0.98696 e; 0.48844 b, 0.48852 b, 0.98696 e, 0.98696 e 5.10730 n
100	78.9325	2.39446 b, 2.39893 b; 2.39446 b, 2.39520 b	0.98696 e, 0.98696 e; 0.98696 e, 0.98696 e 5.22238 n	2.39501 b, 2.39520 b, 0.98696 e, 0.98696 e; 2.39444 b, 2.39449 b, 0.98696 e, 0.98696 e 5.22238 n
110	86.7865 1. 1.	2.63266 b, 2.63594 b; 2.63266 b, 2.63267 b	0.98696 e, 0.98696 e; 0.98696 e, 0.98696 e 5.22705 n	2.63265 b, 2.63265 b, 0.98696 e, 0.98696 e; 2.63265 b, 2.63265 b, 0.98696 e, 0.98696 e 5.22705 n
120	94.6405 1. 1.	2.87116 b, 2.87141 b; 2.87090 b, 2.87090 b	0.98696 e, 0.98696 e; 0.98696 e, 0.98696 e 5.23081 n	2.87090 b 2.87090 b, 0.98696 e, 0.98696 e; 2.87098 b, 2.87098 b, 0.98696 e, 0.98696 e 5.23081 n
125	98.5675 1. 1.	3.02361 b, 3.02867 b; 3.02015 b, 3.02015 b	0.98693 e, 0.98698 e; 0.98696 e, 0.98696 e 5.23241 n	3.02015 b, 3.02015 b, 0.98696 e, 0.98696 e; 3.02015 b, 3.02015 b, 0.98696 e, 0.98696 e 5.23241 n
130	102.494 1. 1.	3.14253 b,3.14747 b; 3.14220 b, 3.14220 b	0.98696 e, 0.98696 e; 0.98696 e, 0.98696 e 5.23386 n	3.14020 b, 3.14020 b, 0.98696 e, 0.98696 e; 3.14220 b, 3.14220 b, 0.98696 e, 0.98696 e 5.23386 n

Table 2. Characteristics of eigenfrequencies for predominantly bending, predominantly planar and nonsymmetrical vibrations of a cylindrical boron plastic panel with rigid-clamped and hinged ends, when $l = 4, s = 5.00326, s = 15.0893$.

m	θ_m^2 $\beta_m'^2$ $\beta_m''^2$	$\eta_{1m} = \eta_{2m}$=0, $\eta_{3m} = \eta_m$. $s = 5.00326$; s=15.0893.	$\eta_{1m} = \eta_{2m}$=η_m, $\eta_{3m} =$ 0. $s = 5.00326$; s=15.0893.	$\eta_{1m} = \eta_{2m}$=$\eta_{3m} = \eta_m$. $s = 5.00326$; s=15.0893.
1	0.98165 0.74668 1.33930	0.03344 b, 0.03395 b; 0.03252 b, 0.03252 b.	5.96781 n	0.03227 b, 0.03281 b; 0.03252 b, 0.03252 b, 5.96781 n
2	1.76715 0.85854 1.16477	0.05562 b, 0.05563 b; 0.05495 b, 0.05499 b.	5.53245 n	0.05561 b, 0.05563 b; 0.05413 b, 0.05413 b, 5.53245 n
3	2.55255 0.90206 1.10857	0.07831 b, 0.07865 b; 0.07810 b, 0.07810 b.	5.38395	0.07681 b, 0.07682 b; 0.07681 b, 0.07681 b, 5.38395 n
4	3.33795 0.92511 1.08095	0.10172 b, 0.10288 b; 0.10160 b, 0.10160 b.	5.30954 n	0.09985 b, 0.09986 b; 0.09985 b, 0.09985 b, 5.30954 n
5	4.12335 0.93937 1.06454	0.12537 b, 0.12569 b; 0.12524 b, 0.12524 b.	-, -; 0.95910 e, 0.95913 e. 5.26510 n	0.12293 b, 0.12294 b; -, -; 0.12293 b, 0.12293 b, 0.95835 e, 0.95842 e. 5.26510 n
16	12.7627 1. 1.	0.38716 b, 0.38722 b; 0.38559 b, 0.38559 b.	0.98695 e, 0.98695 e; 0.98695e, 0.998695 e. 5.09854 n	0.38725 b, 0.38725 b, 0.98696 e, 0.98696 e; 0.38724 b, 0.38724 b, 0.98696 e, 0.98696 e. 5.09854 n
17	13.5481 1. 1.	0.41098 b, 0.41104 b; 0.40918 b, 0.40918 b.	0.98696 e, 0.98696 e; 0.98696 e, 0.98696 e. 5.10050 n	0.41171 b, 0.41171 b, 0.98696 e, 0.98696 e; 0.41152 b, 0.41152 b, 0.98696 e, 0.98696 e. 5.10050 n
18	14.3335 1. 1.	0.43480 b, 0.43486 b; 0.43272 b, 0.43272 b.	0.98696 e, 0.98696 e; 0.98696 e, 0.98696 e. 5.10462 n	0.43602 b, 0.43602 b, 0.98696 e, 0.98696 e; 0.43481 b, 0.43484 b, 0.98696 e, 0.98696 e. 5.10462 n
19	15.1189 1. 1.	0.45863 b, 0.45868 b; 0.45619 b, 0.45619 b.	0.98696 e, 0.98696 e; 0.98696 e, 0.98696 e. 5.10676 n	0.45976 b, 0.45976 b, 0.98696 e, 0.98696 e; 0.45866 b, 0.45874 b, 0.98696 e, 0.98696 e. 5.10676 n
20	15.9087 1. 1.	0.48246 b, 0.48251 b; 0.47957 b, 0.47957 b	0.98696 e, 0.98696 e; 0.98696 e, 0.98696 e.	0.48272 b, 0.48272 b, 0.98696 e, 0.98696 e; 0.48246 b, 0.48246 b, 0.98696 e, 0.98696 e.

			5.17992 n	5.17992 n
100	78.7362 1. 1.	2.38970 b, 2.38996 b; 2.38849 b, 2.38851 b.	 0.98696 e, 0.98696 e; 0.98696 e, 0.98696 e 5.22225 n	2.38770 b, 2.38770 b, 0.98696 e, 0.98698 e; 2.38850 b, 2.38854 b, 0.98696 e, 0.98696 e 5.22225 n
110	86.5901 1. 1.	2.62673 b, 2.62679 b; 2.62672 b, 2.62675 b	 0.98696 e, 0.98696 e; 0.98696 e, 0.98696 e 5.22695 n	2.62783 b, 2.62783 b, 0.98696 e, 0.98696 e; 2.62685 b, 2.62691 b, 0.98696 e, 0.98696 e 5.22695 n
120	94.4441 1. 1.	2.86494 b, 2.86494 b; 2.86496 b, 286496 b	 0.98696 e, 0.98696 e; 0.98696 e, 0.98696 e 5.23072 n	2.86363 b 2.86363 b, 0.98696 e, 0.98696 e; 2.86470 b, 2.86499 b, 0.98696 e, 0.98696 e 5.23072 n
125	98.3711 1. 1.	2.98411 b, 2.98416 b; 2.98408 b, 2.98408 b.	 0.98696 e, 0.98696 e; 0.98696 e, 0.98696 e. 5.23234 n	2.98263 b, 2.98263 b, 0.98696 e, 0.98696 e; 2.98249 b, 2.98249 b, 0.98696 e, 0.98696 e 5.23234 n
130	102.298 1. 1.	3.10323 b, 3.10327 b; 3.10320 b, 3.10320 b.	 0.98696 e, 0.98696 e; 0.98696 e, 0.98696 e 5.23379 n	3.10323 b, 3.10323 b, 0.98696 e, 0.98696 e; 3.10322 b, 3.10322 b, 0.98696 e, 0.98696 e 5.23379 n

Table 3. Characteristics of eigenfrequencies for predominantly bending, predominantly planar and nonsymmetrical vibrations of a cylindrical boron plastic panel with hinged ends, when $l = 4, s = 5.00326, s = 15.0893$.

m	θ_m^3 $\beta_m'^3$ $\beta_m''^3$	$\eta_{1m} = \eta_{2m}$=0, $\eta_{3m} = \eta_m$. $s = 5.00326$; **s=15.0893.**	$\eta_{1m} = \eta_{2m}$=η_m, $\eta_{3m} = 0$. $s = 5.00326$; **s=15.0893.**	$\eta_{1m} = \eta_{2m}$=$\eta_{3m} = \eta_m$. $s = 5.00326$; **s=15.0893.**
1	0.78540 1. 1.	0.02758 b, 0.02853 b; 0.02801 b, 0.02801 b	 5.08131 n	0.02758 b, 0.02852b; 0.02801 b, 0.02802b, 5.08131 n
2	1.57080 1. 1.	0.05189 b, 0.05525 b; 0.05217 b, 0.05304 b	 5.08153 n	0.05189 b, 0.05525 b; 0.05217 b, 0.05304 b, 5.08153 n
3	2.35619 1. 1.	0.07224 b, 0.07241 b; 0.07261 b, 0.07305 b	 5.08189 n	0.07224 b, 0.07241 b; 0.07261 b, 0.07305 b, 5.08189 n
4	3.14159 1. 1.	0.09850 b, 0.10106 b; 0.09570 b, 0.09623 b	 5.08239 n	0.09850 b, 0.10106 b; 0.09571 b, 0.09623 b, 5.08239 n
5	3.92699 1. 1.	0.11932 b, 0.12028 b; 0.11934 b, 0.11977 b	 -, -; 0.98689 e, 0.98692 e. 5.08303 n	0.11932 b, 0.12028 b, -, -; 0.11934 b, 0.11977 b, 0.98694 e, 0.98697 e. 5.08303 n

16	12.5664 1. 1.	0.38120 b, 0.38245 b; 0.38121 b, 0.38207 b	0.98695 e, 0.98696 e; 0.98696 e, 0.98696 e 5.09806 n	0.38120 b, 0.38190 b, 0.98695 e, 0.98696 e; 0.38121 b, 0.38134 b, 0.98696 e, 0.98696 e 5.09806 n
17	13.3518 1. 1.	0.40568 b, 0.40620 b; 0.40569 b, 0.40663 b	0.98696 e, 0.98696 e; 0.98696 e, 0.98696 e. 5.10000 n	0.40568 b, 0.40620 b, 0.98696 e, 0.98696 e; 0.40503 b, 0.40506 b, 0.98696 e, 0.98696 e. 5.10000 n
18	14.1372 1. 1.	0.42885 b, 0.42947 b; 0.42885 b, 0.42947 b	0.98695 e, 0.98696 e; 0.98696 e, 0.98696 e 5.10201 n	0.42885 b, 0.42947 b, 0.98696 e, 0.98696 e; 0.42892 b, 0.42897 b, 0.98696 e, 0.98696 e. 5.10201 n
19	14.9226 1. 1.	0.45267 b, 0.45326 b; 0.45267 b, 0.45279 b	0.98695 e, 0.98696 e; 0.98696 e, 0.98696 e 5.10409 n	0.45267 b, 0.45326 b, 0.98696 e, 0.98696 e; 0.45268 b, 0.45270 b, 0.98696 e, 0.98696 e. 5.10409 n
20	15.708 1. 1.	0.47650 b, 0.47705 b; 0.47650 b, 0.47719 b	0.98695 e, 0.98696 e; 0.98696 e, 0.98696 e. 5.10622 n	0.47650 b, 0.47705 b, 0.98696 e, 0.98696 e; 0.47660 b, 0.47667 b, 0.98696 e, 0.98696 e. 5.10622 n
100	78.5398 1. 1.	2.38310 b, 2.38329 b; 2.38289 b, 2.38329 b	0.98696 e, 0.98696 e; 0.98696 e, 0.98696 e. 5.22212 n	2.38294 b, 2.38310 b, 0.98696 e, 0.98696 e; 2.38242 b, 2.38254 b, 0.98696 e, 0.98696 e. 5.22212 n
110	86.3938 1. 1.	2.62129 b, 2.62188 b; 2.62087 b, 2.62159 b	0.98696 e, 0.98696 e; 0.98696 e, 0.98696 e. 5.22684 n	2.62102 b, 2.62115 b, 0.98696 e, 0.98696 e; 2.62078 b, 2.62081 b, 0.98696 e, 0.98696 e. 5.22684 n
120	94.2478 1. 1.	2.86049 b, 2.86167 b; 2.86049 b, 2.86188 b	0.98696 e, 0.98696 e; 0.98696 e, 0.98696 e. 5.23064 n	2.86003 b 2.86025 b, 0.98696 e, 0.98696 e; 2.86003 b, 2.86010 b, 0.98696 e, 0.98696 e. 5.23064 n
125	98.1748 1. 1.	2.98037 b, 2.98069 b; 2.97940 b, 2.97989 b	0.98696e, 0.98696 e; 0.98696 e, 0.98696 e. 5.23226 n	2.98037 b, 2.98069 b, 0.98696 e, 0.98696 e; 2.97947 b, 2.97955 b, 0.98696 e, 0.98696 e. 5.23226 n
130	102.1020 1. 1.	3.10074 b, 3.10194 b; 3.09760 b, 3.0059 b	0.98696e, 0.98696 e; 0.98696 e, 0.98696 e. 5.23373 n	3.10074 b, 3.10152 b, 0.98696 e, 0.98696 e; 3.09040 b, 3.09261 b, 0.98696 e, 0.98696 e. 5.23373 n

In Tables 1, 2, 3 after the characteristics of eigenfrequencies the type of vibration is indicated: $''b''$- predominantly bending, $''e''$- predominantly planar, and $''n''$ for the new type of vibrations. The elasticity modules $E_1$and E_2 correspond to the directions of generatrix and directrix, respectively. The case with $\eta_{1m} = \eta_{2m} = \eta_{3m} = \eta_m$ corresponds to the problems (1.1)- (1.6). The case with $\eta_{1m} = \eta_{2m} = 0$ and $\eta_{3m} = \eta_m$ corresponds to the problems (1.1)- (1.6), where are no tangential components of the inertia force, i.e., we have the predominantly bending type of vibrations. The case with $\eta_{1m} = \eta_{2m} = \eta_m$, $\eta_{3m} = 0$ corresponds to the predominantly planar type of vibrations.
The following equalities hold for isotropic materials:

$$\frac{B_{12}}{B_{11}} = \frac{B_{12}}{B_{22}} = \sigma, \frac{B_{66}}{B_{11}} = \frac{B_{66}}{B_{22}} = \frac{1-\sigma}{2}. \tag{6.2}$$

Hence, in the dispersion equations and for the calculation characteristics it can be set

$$B_{11} = B_{22} = 1, B_{12} = \sigma, B_{66} = \frac{1-\sigma}{2}.$$

Conclusion. Numerical calculations showed that the first eigenfrequencies localized at the free generators of the cylindrical panels with arbitrary fastening of the ends, where the normal component of inertia force is not zero are the frequencies of predominantly bending type. Along with the first frequencies of quasi-transverse vibrations, there are frequencies of undamped quasi-tangential vibrations. With the increase of m, these vibrations become of Rayleigh type. The analysis of numerical data indicates that for $\varepsilon_m \to 0$ free vibrations of cylindrical panels with arbitrary fastening of the ends decompose into quasi-transverse and quasitangential vibrations, and their frequencies tend to the frequencies of a rectangular plate with arbitrary fastening of the opposite sides. Numerical results showed that asymptotic formulas (5.6) of dispersion equation (2.16) and the mechanism presented here are good reference points for finding the eigenfrequencies of the problems (1.1)-(1.6). The first eigenfrequencies of vibrations of cylindrical panels with arbitrary fastening of the ends, depend on the chosen basic functions 4, 5, 6. Note, that the first dimensionless characteristics η_m^1, η_m^2 and η_m^3 of the problems with rigid-clamped ends, rigid-clamped and hinged ends and hinged ends, respectively, satisfy the inequalities $\eta_m^1 \geq \eta_m^2 \geq \eta_m^3$, $m = \overline{1,\infty}$. For $\theta_m \to 0$, the frequencies of vibrations at free generators of a finite cylindrical panel become practically independent of the basic functions and of the boundary conditions on the ends [8-10,19-21].

Appendix. The analytical expressions for m_{ij} are given below:

$$m_{11} = H\mathcal{X}_1^6 + d_1\mathcal{X}_1^4 + d_2\mathcal{X}_1^2 + d_3, m_{12} = H\overline{\overline{f_5}} + d_1\overline{\overline{f_3}} + d_2\overline{\overline{f_1}},$$

$$m_{13} = H\overline{f_4} + d_1\overline{f_2} + d_2, m_{14} = Hf_3 + d_1f_1;$$

$$m_{21} = T\mathcal{X}_1^5 + d_4\mathcal{X}_1^3 + d_5\mathcal{X}_1, m_{22} = T\overline{\overline{f_4}} + d_4\overline{\overline{f_2}} + d_5, m_{23} = T\overline{f_3} + d_4\overline{f_1},\ m_{24} = Tf_2 + d_4;$$

$$m_{31} = F\mathcal{X}_1^6 + d_6\mathcal{X}_1^4 + d_7\mathcal{X}_1^2 + d_8, m_{32} = F\overline{\overline{f_5}} + d_6\overline{\overline{f_3}} + d_7\overline{\overline{f_1}},$$
$$m_{33} = F\overline{f_4} + d_6\overline{f_2} + d_7,\ m_{34} = Ff_3 + d_6f_1;$$

$$m_{41} = F\mathcal{X}_1^7 + 9\mathcal{X}_1^5 + d_{10}\mathcal{X}_1^3 + d_{11}\mathcal{X}_1, m_{42} = F\overline{\overline{f_6}} + d_9\overline{\overline{f_4}} + d_{10}\overline{\overline{f_2}} + d_{11},$$

$m_{43} = F\overline{f_5} + d_9\overline{f_3} + d_{10}\overline{f_1}, m_{44} = Ff_4 + d_9 f_2 + d_{10};$
$m_{i5} = (-1)^{i-1} m_{i1} exp(z_1), m_{i6} = (-1)^{i-1}(m_{i2} exp(z_2) + m_{i1}[z_1 z_2]),$
$m_{i7} = (-1)^{i-1}(m_{i3} exp(z_3) + m_{i2}[z_2 z_3] + m_{i1}[z_1 z_2 z_3]),$
$m_{i8} = (-1)^{i-1}(m_{i4} exp(z_4) + m_{i3}[z_3 z_4] + m_{i2}[z_2 z_3 z_4] + m_{i1}[z_1 z_2 z_3 z_4]),$
$i = \overline{1,4}$;
$m_{5j} = m_{1\,4+j}, m_{5\,4+j} = m_{1\,j};\ m_{6j} = m_{2\,4+j}, m_{6\,4+j} = m_{2\,j};$
$m_{7j} = m_{3\,4+j}, m_{7\,4+j} = m_{3\,j};\ m_{8j} = m_{4\,4+j}, m_{8\,4+j} = m_{4\,j};\ j = \overline{1,4}.$

$$H = -a^2 \frac{B_{22}}{B_{11}};\ T = a^2 \frac{B_{12} B_{22}}{B_{11} B_{66}};\ F = \frac{B_{22}}{B_{11}}. \tag{7.1}$$

$$d_1 = a^2 \left(\frac{B_{11} B_{11} \beta_m'' - B_{12}^2 \beta_m' + 4 B_{66}^2 \beta_m'}{B_{11} B_{66}} - \frac{B_{22}}{B_{11}} (\boldsymbol{\eta}_{1m}^2 + \varepsilon_m^2) \right),$$

$$d_2 = -\frac{B_{66}}{B_{11}} \boldsymbol{\eta}_{2m}^2 + a^2 \beta_m' \frac{B_{12} + 4B_{66}}{B_{22}} \left(\frac{B_{22}}{B_{11}} \boldsymbol{\eta}_{1m}^2 - B_1 \right) - \varepsilon_m^2 a^2 \left(\frac{B_{22}}{B_{11}} \boldsymbol{\eta}_{1m}^2 - \frac{B_{11} B_{11} \beta_m'' - B_{12}^2 \beta_m' + 4 B_{66}^2 \beta_m'}{B_{11} B_{22}} \right),$$

$$d_3 = (\delta_m \beta_m' - \boldsymbol{\eta}_{2m}^2) \left(\frac{B_{11} B_{11} \beta_m'' - B_{12}^2 \beta_m'}{B_{11} B_{22}} - \frac{B_{66}}{B_{11}} \boldsymbol{\eta}_{1m}^2 \right), \delta_m = 1 + 4a^2 \varepsilon_m^2,$$

$$d_4 = a^2 \left(\frac{B_{12} B_{22}}{B_{11} B_{66}} \varepsilon_m^2 + B_2 - \frac{2B_{12}}{B_{11}} \beta_m' - \frac{B_{12}}{B_{11}} \boldsymbol{\eta}_{1m}^2 \right),$$

$$d_5 = \frac{B_{22}}{B_{11}} \boldsymbol{\eta}_{1m}^2 + \frac{B_{12}}{B_{11}} \boldsymbol{\eta}_{2m}^2 - \frac{B_{11} B_{11} \beta_m'' - B_{12}^2 \beta_m'}{B_{11} B_{66}} + a^2 \beta_m' \frac{B_{12} + 4B_{66}}{B_{22}} \left(\frac{B_{22}}{B_{11}} \boldsymbol{\eta}_{1m}^2 - \frac{B_{22}}{B_{66}} \beta_m'' \right) - \frac{4B_{12}}{B_{11}} \beta_m' a^2 \varepsilon_m^2,\ d_6 = \frac{B_{22}}{B_{11}} \boldsymbol{\eta}_{1m}^2 + \frac{B_{66}}{B_{11}} \boldsymbol{\eta}_{2m}^2 + \frac{B_{22}}{B_{11}} \varepsilon_m^2 - B_1,$$

$$d_7 = (\boldsymbol{\eta}_{2m}^2 - \beta_m') \left(\frac{B_{22}}{B_{11}} \boldsymbol{\eta}_{1m}^2 - \beta_m'' \right) + \frac{B_{12}}{B_{22}} \beta_m' \left(B_2 - \frac{B_{22}}{B_{11}} \boldsymbol{\eta}_{1m}^2 - \frac{B_{66}}{B_{11}} \boldsymbol{\eta}_{2m}^2 \right) + \varepsilon_m^2 \left(4a^2 \frac{B_{12} B_{66}}{B_{11} B_{22}} (\beta_m')^2 - B_1 + \frac{B_{22}}{B_{11}} \boldsymbol{\eta}_{1m}^2 \right),$$

$$d_8 = \frac{B_{12}}{B_{22}} \beta_m' (\boldsymbol{\eta}_{2m}^2 - \delta_m \beta_m') \left(\frac{B_{66}}{B_{11}} \boldsymbol{\eta}_{1m}^2 - \beta_m'' \right),$$

$$d_9 = \frac{B_{22}}{B_{11}} \boldsymbol{\eta}_{1m}^2 + \frac{B_{66}}{B_{11}} \boldsymbol{\eta}_{2m}^2 + \frac{B_{22}}{B_{11}} \varepsilon_m^2 - B_1 - \frac{4B_{66}}{B_{11}} \beta_m',$$

$$d_{10} = (\boldsymbol{\eta}_{2m}^2 - \beta_m') \left(\frac{B_{66}}{B_{11}} \boldsymbol{\eta}_{1m}^2 - \beta_m'' \right) + \frac{B_{12} + 4B_{66}}{B_{22}} \beta_m' \left(B_2 - \frac{B_{22}}{B_{11}} \boldsymbol{\eta}_{1m}^2 - \frac{B_{66}}{B_{11}} \boldsymbol{\eta}_{2m}^2 \right) - \varepsilon_m^2 \left(B_1 + 4 \frac{B_{66}}{B_{11}} \beta_m' - \frac{B_{22}}{B_{11}} \boldsymbol{\eta}_{1m}^2 \right),$$

$$d_{11} = 4\varepsilon_m^2 \beta_m' \left(\frac{B_{66}}{B_{22}} B_1 - \frac{B_{66}}{B_{11}} \boldsymbol{\eta}_{1m}^2 \right) - \frac{B_{12} + 4B_{66}}{B_{22}} \beta_m' (\boldsymbol{\eta}_{2m}^2 - \beta_m') \left(\frac{B_{66}}{B_{11}} \boldsymbol{\eta}_{1m}^2 - \beta_m'' \right).$$

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