ՀԱՅԱՍՏԱՆԻ ԳԻՏՈՒԹՅՈՒՆՆԵՐԻ ԱԶԳԱՅԻՆ ԱԿԱԴԵՄԻԱՅԻ ՏԵՂԵԿԱԳԻՐ
ИЗВЕСТИЯ НАЦИОНАЛЬНОЙ АКАДЕМИИ НАУК АРМЕНИИ

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Shear elastic waves in bi-material multi-layered waveguide

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Key words: shear elastic waves, multilayer waveguide, matrix method, localised waves.

This analytical study demonstrates shear elastic wave propagation in stratified waveguide with emphasis on wave localisation effects using the propagator matrix method. The stratified waveguide consists of two-phase piecewise homogeneous periodically arranged finite number sub-layers along waveguide thickness. Analytical solutions are carried out for traction free waveguide. The existence of two modal types of guided waves is established: i) a localised surface mode occurring in "stopband" and ii) normal modes arising in a "passband" of frequencies.

Сдвиговые упругие волны в многослойном двухфазном волноводе

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Ключевые слова: поперечные упругие волны, многослойный волновод, матричный метод, локализованные волны.

Даное аналитическое исследование посвящено вопросу распространения сдвиговой упругой волны в слоистом волноводе с акцентом на эффекты локализации волн с использованием метода трансфер матриц. Многослойный волновод состоит из конечного числа двухфазных кусочно-однородных периодически распределенных вдоль толщины волновода подслоев. Получены аналитические решения для волновода, поверхности которого свободны от механических напряжений. Установлено существование двух различных мод направленных волн: 1) локализованной поверхностной моды, возникающей в «полосе задерживания» и 2) нормальной моды, возникающей в «полосе пропускания» частот.

Սահքի առաձգական ալիքները բազմաշերտ երկփուլային ալիքատարում

Ղազարյան Կ., Ղազարյան Ռ., Թերզյան Ս.

Հիմնաբառեր՝. լայնական առաձգական ալիքներ, բազմաշերտ ալիքատար, մատրիցների մեթոդ, տեղայնացված ալիքներ:

Տվյալ վերլուծական հետազոտությունը նվիրված է շերտավոր ալիքատարում սահքի առաձգական ալիքի տարածմանը՝ մատրիցների մեթոդի կիրառմամբ, շեշտը դնելով ալիքների տեղայնացման երևույթին: Բազմաշերտ ալիքատարը բաղկացած է կտոր առ կտոր համասեռ, ալիքատարի հաստության երկայնքով պարբերաբար դասավորված վերջավոր թվով ենթաշերտերից: Ստացվել են անալիտիկ լուծումներ մեխանիկական լարումներից ազատ մակերևույթ ունեցող ալիքատարի համար: Հաստատվել է ուղղորդված ալիքների երկու տարբեր մոդերի գոյությունը. 1) տեղայնացված մակերևութային մոդ, որն առաջանում է հաճախությունների «հապաղման շերտում», և 2) նորմալ մոդ, որն առաջանում է հաճախությունների «բաց թողնման շերտում»:

Introduction

Recently much attention has been given to the propagation of elastic waves, called Floquet-Bloch waves, which occurs in elastic periodic structures (phonon crystals) and consisting of an arrangement of coupled substructures with highly contrasting mechanical properties (elastic stiffness and/or mass density). The most notable feature of phonon crystals is the existence of finite "stopband" of frequencies in which elastic waves are unable propagate in any direction [1-5]. This important feature makes possible to use such crystals in the design of novel engineering material and structures.

The existence of a new type of surface SH wave which propagates along the free surface of a periodically layered half-space was first demonstrated by Auld *et al*. [6]. They have shown that harmonic wave can be exponentially attenuated in periodically layered half-space in a finite "stopband" of frequencies. The detailed analysis of these SH waves was given by Camley *et al.* [7] and further developed by Chen *et al.* and Jorge *et al.*, which studies are reported in [8,9], respectively. *Boudouti et al.* [10] studied the transverse surface elastic wave in the semi-infinite N-layer super-lattices created by periodic repetition of N different elastic slabs and the special case of a four-layer super-lattice was considered. *Shuvalov et al.* [11] considered the SH surface wave in a periodically layered semi space with an arbitrary non-homogeneous unit cell profile. The study has shown that the existence and spectral properties of the SH surface wave is directly related to geometry and physical properties.

An extensive overview of historical developments with an in-depth literature (more than 400 references) and technical review of recent progress in the field of dynamics problem of elastic and acoustic wave motion in periodic structures is given by Hussein [12]. A review of the most widely-used methods determining structure of eigenmodes propagating in periodic materials was presented by *Gazalet et al* [13]. In one of the most recent papers on waves in periodic structures *Shmuel and Band* [14] shows that the frequency spectrum of periodic 1D two-phase laminates has a universal structure, independent of the geometry of their unit-cell and specific physical properties.

Vibration problems of finite periodic structures are closely related to the problems of wave reflection and transmission by a finite periodic layer. The reflection and transmission of electromagnetic waves through periodically stratified medium was considered by *Yeh et al.* [15], where the analytical expression of the reflectivity of a finite multilayer two phase dielectric reflector was presented. For semi-infinite periodic multilayer dielectric medium, consisting of alternating layers of different indices of refraction, the existence of surface electromagnetic wave was shown in finite "stopband" of frequencies. In the framework of matrix analysis the implications of the band structure of an infinite periodic structure for wave reflection by a finite structure are demonstrated by *Lekner* [16] for electromagnetic waves in stratified dielectric media. Numerous problems of wave propagation in elastic multilayered medium were considered by *Brekhovskikh* [17].

Shear wave transmission characteristics in elastic media that have periodic microstructure over a finite spatial length were examined by *Kobayashi et al.* [18] for two classes of such media, namely, one-dimensional multilayered media with finite-length periodicity and two-dimensional composite media with square arrays of aligned fibers within a finite length.

A few recently published studies were devoted to vibration problems of finite 1D periodic rods and beams. In the framework of Galerkin method *Ying* and *Ni [*19, 20] considered the vibration of finite length beam with arbitrary periodic modulation of beam rigidity and cross-section parameters. By means of numerical analysis the relationship between the natural frequencies of the non-uniform beams with finite periodicity and the band gap boundaries of the corresponding infinite periodic beam was investigated. *Xu et al.* [21] employed the transfer matrix method to study the natural frequencies of the two-phase beams with modulation by finite periodic uniform cells. The effects of the amounts, cross section ratios, and arrangement forms of the periodic cells on the natural frequencies were explored and the relationship between the natural bending frequencies of the beams with finite periodicity and the band gap boundaries of the corresponding infinite periodic beam has been discussed. Based on physical models of elastic rod and beam *Hvatov* and *Sorokin.* [22, 23] compared the eigenfrequency spectra of finite periodic structures with the location of the “stopband” for their infinite counterparts. Special attention was paid to eigenfrequencies and eigenmodes of a single periodicity cell with appropriate boundary conditions. The influence of the number of periodicity cells in a finite multi-layered structure on its eigenfrequency spectrum was analysed.

Elastic wave localisation in waveguide caused by micropolarity properties was shown by *Ambartsumian, Avetisyan and Belubekyan.* [24]. *Avetisyan et al.* [25, 26] have shown that an unevenness of waveguide walls can be the reason of shear wave localisation in layered elastic and piezoelectric composites.

To the best of the authors’ knowledge less attention was paid to wave localization problem in finite periodically arranged structure. Hence, in the present paper, using the well-known propagator matrix formalism suggested by *Gilbert* and *Backus,* [27] and developed by *Alshits et al.* [28], an analytical formulation is provided for shear elastic wave propagation in periodically stratified layers with emphasis on wave localisation effects. The stratified finite layer consists from two-phase piecewise homogeneous sub-layers or functionally graded elastic alternating sub-layers when layer plane surfaces are free from mechanical tractions.

Multi-layered piecewise homogeneous waveguide

Let’s consider shear waves propagating along a multi-layered elastic waveguide constituted by a finite number of repeated different two sub-layers consisting from different elastic materials A and B, see Fig. 1. Each of these sub-layers of widths d_1, d_2 , is labelled by the index $(s)=1,\ (s)=2$ within the unit cell labelled by the index $\mathrm{n}\,(\mathrm{n}=1,2,3...\mathrm{N})$. Each of the two sub-layers is assumed to be perfectly bonded to the adjoining sub-layers. The layer extends from the top surface $x=0$ to the bottom surface $x=Nd$, and $d=d_1+d_2$, N is the number of elementary units.

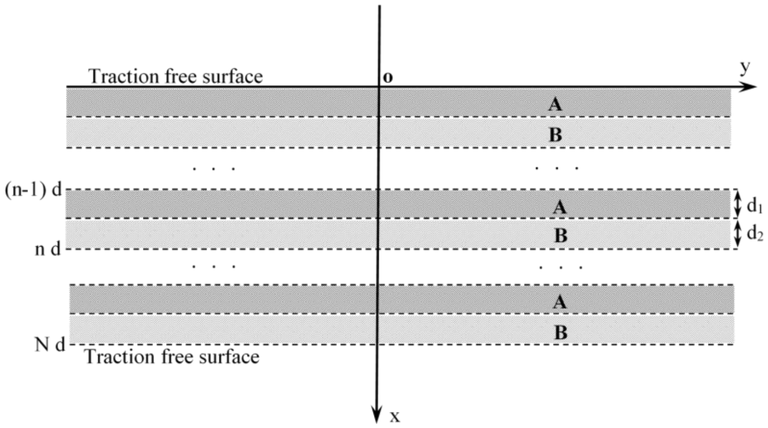


Figure 1: Geometry of the multi-layered elastic waveguide constituted by a finite number of repeated different elastic sub-layers A, B

The elastic displacements and stresses obey to the anti-plane equations of motion and Hooke's law. Choosing the anti-plane deformation in the z -direction one has

$$\partial_x \sigma_{xz} + \partial_y \sigma_{yz} = \rho \partial_{t,t} u_z \, , \; \sigma_{xz} = \mu \partial_x u_z , \sigma_{yz} = \mu \partial_y u_z \tag{1}$$

where $u_z(x, y, t)$ is the displacement in z - direction.

Considering a steady SH-wave propagating with

$$u_z(x, y, t) = u(x) \exp[i(ky - \omega t)] \tag{2}$$

where k, ω are wave number and frequency, the solutions for functions $u_n^{(s)}(x)$ within each material A, B domains of the sub-layer material can be found as the sum of incident and reflected plane waves

$$u_n^{(s)}(x) = \alpha_{ni}^{(s)} \exp(iq_s x) + \alpha_{nr}^{(s)} \exp(-iq_s x) \tag{3}$$

Here $q_s = \sqrt{\omega^2/c_s^2 - k^2}, c_s^2 = \mu_s/\rho_s$, and correspondingly, $\alpha_{ni}^{(s)}$, $\alpha_{nr}^{(s)}$ are the complex amplitudes of the incident and reflected plane waves, respectively.

According to Eq. (2) one can define $\sigma_{zxn}^{(s)}$ as

$$\sigma_{zxn}^{(s)} = \tau_n^{(s)}(x) \exp[i(ky - \omega t)] \, , \tag{4}$$

$$\tau_n^{(s)}(x) = i\mu_s q_s \left[\alpha_{ni}^{(s)} \exp(iq_s x) - \alpha_{nr}^{(s)} \exp(-iq_s x) \right] \tag{5}$$

Enforcing the continuity of tractions and displacement jump boundary conditions at the interfaces of two materials that is

$$\begin{gathered} u_n^{(1)}(x) = u_n^{(2)}(x), \tau_n^{(1)}(x) = \tau_n^{(2)}(x); \quad x = (n-1)d + d_1 \\ u_n^{(2)}(x) = u_{n+1}^{(1)}(x), \tau_n^{(2)}(x) = \tau_{n+1}^{(1)}(x); \quad \mathrm{x} = \mathrm{nd} \\ \mathrm{n} = 1, 2, ..., N \end{gathered} \tag{6}$$

Since the interface continuity conditions are imposed on functions $u_n^{(s)}(x), \tau_n^{(s)}(x)$ it is convenient to introduce the following column field vectors

$$\bar{U}_n^{(s)}(x) = \begin{pmatrix} u_n^{(s)} \\ \tau_n^{(s)} \end{pmatrix}, \; \bar{A}_n^{(s)} = \begin{pmatrix} \alpha_{ni}^{(s)} \\ \alpha_{nr}^{(s)} \end{pmatrix}, \tag{7}$$

In matrix form the solutions Eqs.(3-,5) can be cast as

$$\bar{U}_n^{(s)}(x) = \hat{\mathrm{F}}_n^{(s)}(x) \cdot \bar{A}_n^{(s)}, \tag{8}$$

where

$$\hat{\mathrm{F}}_n^{(s)}(x) = \begin{pmatrix} \exp(\mathrm{i}q_s x), & \exp(\text{-i}q_s x) \\ \mathrm{i}\mu_s q_s \exp(\mathrm{i}q_s x), & -\mathrm{i}\mu_s q_s \exp(\text{-i}q_s x) \end{pmatrix} \tag{9}$$

Let note that the transmission conditions reported in Eqs. (6) lead to the conditions of continuities of the field vectors $\bar{U}_n^{(s)}(x)$ at separation interfaces of the sub-layers.

Statement of the problem, propagator matrix method

With the view of linking the field values of the vectors $\bar{U}_1^{(1)}(x), \bar{U}_N^{(2)}(x)$, between top $x=0$ and bottom $x=Nd$ surfaces of the waveguide, a propagator matrix method [26] will be used.

The method considers two neighbouring points $x_{1n}^{(s)}, x_{2n}^{(s)}$ within each material in domains of the sub-layers $A, B,$ of the n-th cell. For values of field vectors $\bar{U}_n^{(s)}(x)$ in these points the following conditions hold valid

$$\bar{U}_n^{(s)}\left(x_{1n}^{(s)}\right) = \hat{\mathrm{F}}_n^{(s)}\left(x_{1n}^{(s)}\right)\cdot \bar{A}_n^{(s)}, \quad \bar{U}_n^{(s)}\left(x_{2n}^{(s)}\right) = \hat{\mathrm{F}}_n^{(s)}\left(x_{2n}^{(s)}\right)\cdot \bar{A}_n^{(s)} \tag{10}$$

Eliminating vectors $\bar{A}_n^{(s)}$ from Eq. (10) the relation linking $\bar{U}_n^{(s)}$ vector field values within each material can be found. This is:

$$\bar{U}_n^{(s)}\left(x_{2n}^{(s)}\right) = T_n^{(s)}\left(x_{1n}^{(s)}, x_{2n}^{(s)}\right)\bar{U}_n^{(s)}\left(x_{1n}^{(s)}\right), \tag{11}$$

Herein $\hat{T}_n^{(s)}\left(x_{1n}^{(s)}, x_{2n}^{(s)}\right) = \hat{\mathrm{F}}_n^{(s)}\left(x_{2n}^{(s)}\right)\left(\hat{\mathrm{F}}_n^{(s)}\left(x_{1n}^{(s)}\right)\right)^{-1}$ is the transfer matrix in each sub-layer.

$$T_n^{(s)}\left(x_{1n}^{(s)}, x_{2n}^{(s)}\right) = \begin{pmatrix} \cos\left(q_s\left(x_{2n}^{(s)} - x_{1n}^{(s)}\right)\right) & \left(\mu_s q_s\right)^{-1}\sin\left(q_s\left(x_{2n}^{(s)} - x_{1n}^{(s)}\right)\right) \\ -\mu_s q_s \sin\left(q_s\left(x_{2n}^{(s)} - x_{1n}^{(s)}\right)\right) & \cos\left(q_s\left(x_{2n}^{(s)} - x_{1n}^{(s)}\right)\right) \end{pmatrix} \tag{12}$$

Let now consider the n-th cell of the structure. Using the continuity conditions of field vectors $\bar{U}_n^{(s)}(x)$ at interfaces $x_0 = (n-1)d + d_1$ one obtains

$$\bar{U}_n^{(1)}(x_0) = \bar{U}_n^{(2)}(x_0), \tag{13}$$

while Eq.(11) leads to the matrix equations

$$\bar{U}_n^{(2)}\left(nd\right) = \hat{M}\bar{U}_n^{(1)}\left((n-1)\,\mathrm{d}\right), \tag{14}$$

where $\hat{M} = \hat{T}_n^{(2)}\left(d_0, nd\right)\hat{T}_n^{(1)}\left((n-1)d, d_0\right), \quad d_0 = (n-1)d + d_1,$ (15)

Herein $\hat{\mathrm{M}}$ is the unimodal propagator matrix for SH wave field, which links the field vectors at the top and bottom of the n-*th* cell.

The explicit expressions of the unimodal propagator matrix $\hat{\mathrm{M}}$ elements can be derived as

$$m_{11}=\cos\left(d_1q_1\right)\cos\left(d_2q_2\right)-\frac{q_1\mu_1\sin\left(d_2q_2\right)\sin\left(d_1q_1\right)}{q_2\mu_2}$$
$$m_{12}=\frac{\cos\left(d_1q_1\right)\sin\left(d_2q_2\right)}{q_2\mu_2}+\frac{\sin\left(d_1q_1\right)\cos\left(d_2q_2\right)}{q_1\mu_1}\qquad(16)$$
$$m_{21}=-q_2\mu_2\cos\left(d_1q_1\right)\sin\left(d_2q_2\right)-q_1\mu_1\cos\left(d_2q_2\right)\sin\left(d_1q_1\right)$$
$$m_{22}=\cos\left(d_1q_1\right)\cos\left(d_2q_2\right)-\frac{q_2\mu_2}{q_1\mu_1}\sin\left(d_1q_1\right)\sin\left(d_2q_2\right)$$

Let note that elements of matrix $\hat{M}$ do not depend of cell number n. Repeating this procedure the n-th times the propagator unimodal matrix $\hat{\mathrm{M}}^n$ can be found. The matrix $\hat{\mathrm{M}}^n$ links the field vectors at $x=0$ and $x=nd$ surfaces of the waveguide.

$$\hat{M}^n\bar{U}_1^{(1)}\left(0\right)=\bar{U}_n^{(2)}\left(nd\right),\qquad n=1,2,....N\qquad(17)$$

According to Sylvester's matrix polynomial theorem [28] for 2x2 matrices the elements of the n-th power of an unimodal matrix $\hat{M}^n$ can be cast as

$$\hat{\mathrm{M}}^n=\begin{pmatrix}M_{11} & M_{12}\\ M_{21} & M_{22}\end{pmatrix}\qquad(18)$$

and can be simplified using the following matrix identity

$$M_{11}=m_{11}S_{n-1}\left(\eta\right)-S_{n-2}\left(\eta\right);\quad M_{12}=m_{12}S_{n-1}\left(\eta\right)$$
$$M_{21}=m_{21}S_{n-1}\left(\eta\right);\qquad M_{22}=m_{22}S_{n-1}\left(\eta\right)-S_{n-2}\left(\eta\right)\qquad(19)$$

where $S_n\left(\eta\right)$ are the Chebyshev polynomials of second kind, namely

$$S_n\left(\eta\right)=\frac{\sin\left((n+1)\phi\right)}{\sin\left(\phi\right)};\cos\left(\phi\right)=\eta;$$
$$\eta=\frac{1}{2}Tr\left(\hat{M}\right)=\frac{1}{2}\left(m_{11}+m_{22}\right);\qquad(20)$$

The first Chebyshev polynomials are

$$S_0\left(\eta\right)=1,\qquad S_1\left(\eta\right)=2\eta;\qquad S_2\left(\eta\right)=4\eta^2-1$$

Subsequent polynomials may be obtained from the recurrence relation of Chebyshev polynomials [29]

$$S_m(\eta)=2\eta S_{m-1}(\eta)-S_{m-2}(\eta)\qquad(21)$$

The matrix trace $\mathrm{Tr}\left(\hat{M}\right)$, namely the condition $\left|\mathrm{Tr}(\hat{M})\right|>2$, defines the “stopband” of frequencies [15], ranges of eigenfrequencies in which waves cannot propagate in the infinite periodic medium consisting of periodically repeated sub-layers of materials A and B. The “stopband” edges are given by $\left|\mathrm{Tr}\left(\hat{M}\right)\right|=2$.

By drawing an analogy with an infinite periodic medium the "stopband" is the ranges of finite layer eigenfrequencies satisfying to condition $|\eta|>1$ and "passband" the ranges where $|\eta|<1$.

Consider now a boundary value problem for the top $x=0$ and bottom $x=Nd$ surface waveguides free from mechanical tractions.

In this case the following matrix equation can be imposed

$$\hat{M}^N\begin{pmatrix} u_1^{(1)}(0) \\ 0 \end{pmatrix} - \begin{pmatrix} u_N^{(2)}(Nd) \\ 0 \end{pmatrix} = 0 \tag{22}$$

From the non-trivial solution of Eq. (22) one can find that $M_{21}=0$ and therefore the two alternative equations must be considered

$$m_{21}(\omega)=0 \tag{23}$$

$$S_{N-1}(\eta(\omega))=0 \tag{24}$$

Alongside Eq. (22) one can consider the matrix equation such as

$$\hat{M}^n\begin{pmatrix} u_1^{(1)}(0) \\ 0 \end{pmatrix} - \begin{pmatrix} u_n^{(2)}(nd) \\ \tau_n^{(2)}(nd) \end{pmatrix} = 0$$

and the relation between field vector values can be found as

$$u_n^{(2)}(nd)=\left(m_{11}S_{n-1}-S_{n-2}\right)u_1^{(1)}(0); \qquad n=1,2,3...N \tag{25}$$

Based on Eqs. (16, 23) we have

$$q_2\mu_2\tan\left(d_2q_2\right)+q_1\mu_1\tan\left(d_1q_1\right)=0; \tag{26}$$

This is the dispersion equation for the single bi-material unit layer, walls of which are free from mechanical tractions. This equation was also discussed in [30].

The roots of Eq. (26) are curves in the phase-plane (ω,k), each point of which corresponds to a wave freely propagating in the waveguide.

From Eq. (23), since the matrix $\hat{M}$ is an unimodular matrix it follows that ;

$$m_{11}(\omega)m_{22}(\omega)=1,\ \eta(\omega)=\frac{1}{2}\left(\gamma+\frac{1}{\gamma}\right) \tag{27}$$

where $\gamma=m_{11}(\omega)$.

Equation (25) can be written now as

$$u_n^{(2)}(nd)=\left(\gamma S_{n-1}(\eta)-S_{n-2}(\eta)\right)u_1^{(1)}(0) \tag{28}$$

Using the recurrence relation $S_n(\eta)=2\eta S_{n-1}(\eta)-S_{n-2}(\eta)$ for the Chebyshev polynomials of the second kind the following relation is obtained

$$\gamma S_n(\eta)-S_{n-1}(\eta)=\gamma\left(\left(\gamma+\gamma^{-1}\right)S_{n-1}(\eta)-S_{n-2}(\eta)\right)-S_{n-1}(\eta)=$$
$$=\gamma\left(\gamma S_{n-1}(\eta)-S_{n-2}(\eta)\right)$$

which can be rewritten as $P_{n+1}(\eta) = \gamma P_n(\eta)$, where $P_n(\eta) = \gamma S_{n-1}(\eta) - S_{n-2}(\eta)$.

Taking into account that $P_1(\eta) = \gamma$, the following identity can be obtained valid for all integers starting from n = 1

$$P_n(\eta) = \gamma^n \tag{29}$$

Hence it follows from (28) that for for the eigenfrequencies satisfying to dispersion equation (26)

$$u_n^{(2)}(nd) = \gamma^n u_1^{(1)}(0), \quad n = 1, 2, ...N \tag{30}$$

In "stopband" range defined by $|\eta| > 1$, for the eigenfrequencies satisfying to dispersion equation (26) in the finite range where $|\gamma| < 1$, the guided wave are localised in the neighbourhood of the waveguide top surface and decayed at the waveguide bottom surface with increasing of cell numbers. If $|\gamma| > 1$ the localisation takes place in the neighbourhood of the bottom surface waveguide.

Similar localisation effects have been obtained in [7-11] for semi-infinite periodic piecewise layered, inhomogeneous structures, where was shown that semi-infinite periodic elastic medium can support propagation of shear surface waves.

Besides of the dispersion equation (26) defining the localisation mode the following dispersion equation should be considered as well

$$S_{N-1}(\eta) = 0 \tag{31}$$

This equation has $(N-1)$ roots on the range $\eta \in (-1, 1)$ and its zeroes are given by

$$\eta_m = \cos\left(m\pi N^{-1}\right), \qquad m = 1, 2.....(N-1) \ . \tag{32}$$

Taking into account that $S_{N-2}(\eta_m) = (-1)^{m+1}$, from the relation Eq. (25), one obtains

$$u_n^{(2)}(nd) = \frac{(-1)^{m+1}}{\sin\left(\frac{m\pi}{N}\right)} \left[m_{11} \sin\left(\frac{m\pi}{N}(N-n)\right) - \sin\left(\frac{m\pi}{N}(N-n+1)\right) \right] u_1^{(1)}(0)$$

$$u_N^{(2)}(Nd) = (-1)^m u_1^{(1)}(0) \tag{33}$$

From Eqs. (32) and (33) one can state that, in the eigenfrequency "passband" ranges $|\eta| < 1$, there exist the $(N-1)$ wave normal modes where guided wave are distributed along the waveguide height according to the first correlation of Eq.(33) and having the same magnitude at the top and the bottom surfaces.

The analysis of this problem has shown that in the stratified waveguide with the piecewise homogeneous finite number unit periodic cells there exit two guided waves modes: 1) a localised mode occurring in the "stopband" frequency range and 2) $(N-1)$ normal modes occurring in the "passband" frequency region.

Results and Discussion

We now illustrate the obtained theoretical results providing analytical and numerical analysis of the equations and relations defining the localized mode of the guided wave in piecewise waveguide. Numerical calculations have been carried out for materials: A made from Cu material with the following properties ($\rho_1 = 8960 kgm^{-3}$, $\mu_1 = 47.7 GPa$), while B made from Al material with the properties ($\rho_2 = 2720 kgm^{-3}$, $\mu_2 = 26.2 \cdot GPa$)

First note that for piecewise waveguide all roots of the dispersion equation Eq. (26) of the localized mode are in the "stopband" range. The equation Eq. (26) has no solution when both $q_1(\omega)$ and $q_2(\omega)$ are imaginary. Other types of solutions are possible: $q_1(\omega)$ imaginary and $q_2(\omega)$ real and visa verse, and $q_1(\omega), q_2(\omega)$ both real.

Since for the given materials $c_1 < c_2$ we consider the specific "stopband" frequency range where $kc_1 < \omega < kc_2$. In this range $q_2(\omega) = iq_{20}(\omega), q_{20}(\omega) = \sqrt{k^2 - \omega^2/c_2^2}$, $q_1(\omega)$ is real and instead of Eq. (30) the following relation can be obtained when ω are roots of the Eq.(26)

$$u_n^{(2)}(nd) = \left(\frac{\cos(d_1 q_1)}{\cosh(d_2 q_{20})} \right)^n u_1^{(1)}(0), \qquad n = 1, 2, ...N$$

On the other hand, using relation Eq.(11,12), within sub-layers of material B we have that

$$u_n^{(2)}(nd) = \frac{1}{\cosh(d_2 q_{20})} u_n^{(2)}((n-1)d + d_1);$$

Therefore in this specific eigen frequencies range, in addition to displacement attenuation due to the cell number n increasing, it is found that the attenuation takes place also in within the material B bodies. This type of localisation was also reported in [30] where the theory of Love waves was generalized to a single bi-material layer consisting from a finite-thickness substrate covered by a finite-thickness slap having a lower shear elastic speed.

For a waveguide for any value of dimensionless wave number $\kappa = kd$ there are infinite number of discrete spectrum of eigenfrequencies which correspond to localised vibrations. Mode of these eigenfrequencies shown in Figure 2 are the solutions of dispersion equations Eq.(26) which in dimensionless notations can be written as

$$\mu\sqrt{\vartheta^2 - \kappa^2} \sin\left(\delta\sqrt{\vartheta^2 - \kappa^2}\right) \cos\left((1-\delta)\sqrt{\beta^2\vartheta^2 - \kappa^2}\right) +$$
$$+\sqrt{\beta^2\vartheta^2 - \kappa^2} \cos\left(\delta\sqrt{\vartheta^2 - \kappa^2}\right) \sin\left((1-\delta)\sqrt{\beta^2\vartheta^2 - \kappa^2}\right) = 0$$
$$\vartheta = \omega d/c_1; \delta = d_1/d; \kappa = kd; \mu = \mu_1/\mu_2; \beta = c_1/c_2 \qquad (34)$$

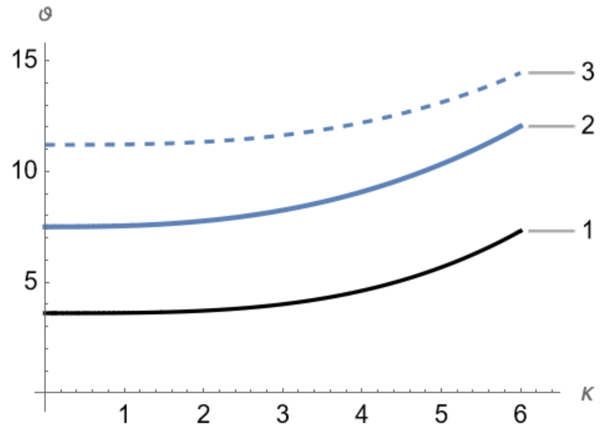
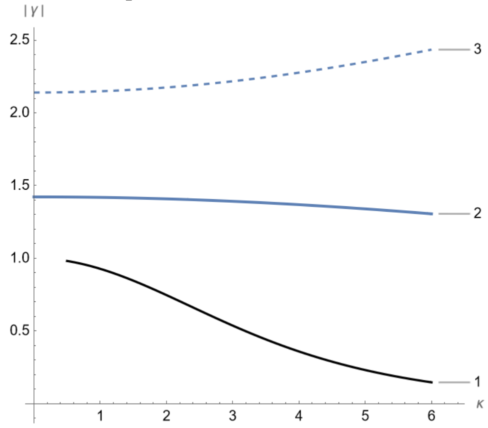


Fig.2 The dispersion curves of the localisation modes

In Figure 3 graphs of localisation coefficient $\left|\gamma(\kappa,\vartheta)\right|$ corresponding to the localised modes are presented.



Fiq.3 Localisation coefficient curves defining the guided wave amplitude localisation

As it follows from Figure 3 the strong localization takes place for the first and third modes. All the curves in Figures 2,3 correspond to the case $\delta = 0.4$.

Conclusions

The paper is devoted to the localisation problem of the shear elastic wave in stratified waveguide with plane surfaces free from mechanical tractions. The stratified waveguide constitutes from a finite number of periodically repeated perfectly bonded sub-layers. The main results of this paper are as follows: first, it is shown that in stratified piecewise bi-material waveguide with surfaces free from mechanical tractions, there exist two modal types of guided waves: i) a localised surface mode occurring in a “stopband” range and ii) normal modes arising in a “passband” range of eigenfrequencies. The guided wave may be localised at the neighbourhood of the waveguide top surface and decays at the waveguide bottom surface with increasing cell numbers; the localisation can take place at the neighbourhood of the bottom surface waveguide and that depends on the eigenfrequencies and mechanical properties of piecewise materials.

This study opens up new opportunities into the use of stratified waveguide in the design of novel engineering material and structures.

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