ՀԱՅԱՍՏԱՆԻԳԻՏՈՒԹՅՈՒՆՆԵՐԻԱԶԳԱՅԻՆԱԿԱԴԵՄԻԱՅԻՏԵՂԵԿԱԳԻՐ
ИЗВЕСТИЯ НАЦИОНАЛЬНОЙ АКАДЕМИИ НАУК АРМЕНИИ

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ADHESIVE INTERACTION OF A PIECEWISE-HOMOGENEOUS ORTHOTROPIC PLATE WITH AN ELASTIC BEAM

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Keywords: contact problem, orthotropic plate, elastic inclusion, integro-differential equation, integral transformation, Riemann problem, asymptotic estimates

Ն.Ն. Շավլակաձե

Կտոր առ կտոր համասեռ օրթոտրոպ սալի սոսնձային փոխազդեցությունը առաձգական հեծանի հետ

Հիմնաբառեր: կոնտակտային խնդիր, օրթոտրոպ սալ, առաձգական ներդրակ, ինտեգրո-դիֆֆե-րենցիալ հավասարում, ինտեգրալ ձևափոխություն, Ռիմանի խնդիր, ասիմպտոտիկ գնահատականներ

Դիտարկված է կտոր առ կտոր համասեռ օրթոտրոպ սալ, որը ուժեղացված է ուղիղ անկյան տակ նյութերի բաժանման գիծ դուրս եկող վերջավոր երկարության սեպաձև ներդրակով և բեռնավորված է շոշոփող և նորմալ ուժերով:Անալիտիկ ֆունկցիաների տեսության մեթոդներով խնդիրը բերված է ֆիկսված եզակիությամբ սինգուլյար ինտեգրո-դիֆֆերենցիալ հավասարումների լուծմանը: Այն դեպքում, երբ ներդրակ-հեծանը ունի միայն ծռման կոշտություն և բեռնավորված է նորմալ ուժերով, ինտեգրալ ձևափոխությունների միջոցով ստացված է Ռիմանի խնդիր, որի լուծումը ներկայացված է բացահայտ տեսքով: Որոշված են կոնտակտային գծի վրա առաջացող նորմալ լարումները և պարզված է կոնտակտային լարումների վարքը եզակի կետերի շրջակայքում:

Н.Н. Шавлакадзе

Адгезионное взаимодействие кусочно-однородной ортотропной пластины с упругой балкой

Ключевые слова: контактная задача, ортотропная пластина, упругое включение, интегро-дифференциальное уравнение, интегральное преобразование, задача Римана, асимптотические оценки

Рассмотрена кусочно-однородная упругая ортотропная пластина, армированная конечным включением клиновидной формы, выходящаяна границу раздела материалов под прямым углом и нагруженная касательными и нормальными силами. С помощью методов теории аналитических функций задача сводится к сингулярным интегро-дифференциальным уравнениям с фиксированной особенностью. Когда включение-балка имеет только изгибную жесткость и нагружена нормальными силами, с помощью интегрального преобразования получается задача Римана, решение которой представлено в явном виде. Определены нормальные контактные напряжения вдоль линии контакта и установлено поведение контактных напряжений в окрестностях особых точек.

A piecewise-homogeneous elastic orthotropic plate, reinforced with a finite inclusion of the wedge-shaped, which meets the interface at a right angle and is loaded with tangential and normal forces is considered. By using methods of the theory of analytic function, the problem is reduced to singular integro-differential equations with fixed singularity. When the inclusion-beam has only bending stiffness and is loaded with normal forces, using an integral transformation a Riemann problem is obtained, the solution of which is presented in explicit form. The normal contact stresses along the contact line are determined and the behavior of the contact stresses in the neighborhood of singular points is established.

Introduction

The solutions of static contact problems for different domains, reinforced with elasticthin inclusions andpatches of variable stiffness were obtained, and the behavior of the contact stresses at the ends of the contact line has been investigated, depending on the geometrical and physical parameters of these thin-walled elements [1-10]. The first fundamental problem for a piecewise-homogeneous plane was solved, when a crack of finite length arrives at the interface of two bodies at the right angle [11], and also a similar problem for a piecewise-homogeneous plane when acted upon by symmetrical normal stresses at the crack sides [12, 13], as well as the contact problems for piecewise-homogeneous planes with a semi-infinite and finite inclusion [14].

Problem statement and its solution

Suppose an elastic body S occupies the plane of a complex variable $z = x + iy$, which contains an elastic patch along the segment $l_1 = (0,1)$ and consists of two half-planes of dissimilar materials

$$S^{(1)} = \{z \mid \operatorname{Re} z > 0, z \notin [0,1]\}, \qquad S^{(2)} = \{z \mid \operatorname{Re} z < 0\}$$

joined along the Oy axis. In particularly, we will consider a piecewise-homogeneous orthotropic plate in the condition of plane deformation, which consists of two half-planes of dissimilar materials and reinforced with a finite patch (inclusion) with modulus of elasticity $E_1(x)$, thickness $h_1(x)$ and Poisson's coefficient ν_1. It is assumed that the horizontal and vertical stresses with intensity $\tau_0(x)$ and $p_0(x)$ acts on the patch along the Ox axis (the functions $\tau_0(x)$ and $p_0(x)$ are bounded functions on the finite interval). The patchin the vertical direction bends like a beam (has a finite bending stiffness) and besides in the horizontal direction the patch compressed or stretched like rod being in uniaxial stress state. The contact between the plate and patch is realized by a thin glue layer with width h_0 and Lame's constants λ_0, μ_0. The contact conditions has the form [15]

$$u_1(x) - u^{(1)}(x,0) = k_0 \tau(x), \; v_1(x) - v^{(1)}(x,0) = m_0 p(x), \qquad 0 < x < 1 \qquad (1.1)$$

where $u^{(1)}(x,y), v^{(1)}(x,y)$ are displacement components of the plate points and $u_1(x), v_1(x)$ displacements of the patch points along the Ox axis, $k_0 = h_0/\mu_0$ and $m_0 = h_0/(\lambda_0 + 2\mu_0)$.

We have to define the law of distribution of tangential $\tau(x)$ and normal $\mathrm{p}(x)$ contact stresses on the line of contact, the asymptotic behavior of these stresses at the ends of the patch.

According to the equilibrium equation of patch elements and Hooke's law we have:

$$\frac{du_1(x)}{dx} = \frac{1}{E(x)} \int_0^x [\tau(t) - \tau_0(t)] dt,$$

$$\frac{d^2}{dx^2} D(x) \frac{d^2 v_1(x)}{dx^2} = p_0(x) - p(x), \quad 0 < x < 1 \tag{1.2}$$

and the equilibrium equation of the patch has the form

$$\int_0^1 [\tau(t) - \tau_0(t)] dt = 0, \quad \int_0^1 [p(t) - p_0(t)] dt = 0, \quad \int_0^1 t[p(t) - p_0(t)] dt = 0, \tag{1.3}$$

where $E(x) = \frac{E_1(x) h_1(x)}{1 - \nu_1^2}$, $\mathrm{D}(x) = \frac{E_1(x) h^3{}_1(x)}{1 - \nu_1^2}$.

At the interface of the two materials we have the continuity conditions

$$\sigma_x^{(1)} = \sigma_x^{(2)}, \qquad \tau_{xy}^{(1)} = \tau_{xy}^{(2)}, \qquad u^{(1)} = u^{(2)}, \qquad v^{(1)} = v^{(2)} \tag{1.4}$$

where $\sigma_x^{(\mathrm{k})}, \tau_{xy}^{(\mathrm{k})}$ are the stress components and $u^{(\mathrm{k})}, v^{(\mathrm{k})}$ are the displacement components.

The boundary conditions ofthe components of the stress and displacement fields in the half-plane $S^{(1)}$ has the form

$$\sigma_y^{(1)+} - \sigma_y^{(1)-} = p(x),\ \tau_{xy}^{(1)+} - \tau_{xy}^{(1)-} = \tau(x),\ u^{(1)+} = u^{(1)-},\ v^{(1)+} = v^{(1)-},\ 0 < x < 1 \tag{1.5}$$

Using Lekhnitskii's formulae [16] the components of stress and displacement are represented in the form

$$\begin{aligned}
\sigma_x^{(k)} &= -2\,\mathrm{Re}[\beta_k^2 \Phi_k(z_k) + \gamma_k^2 \Psi_k(\varsigma_k)] \\
\sigma_y^{(k)} &= 2\,\mathrm{Re}[\Phi_k(z_k) + \Psi_k(\varsigma_k)] \\
\tau_{xy}^{(k)} &= 2\,\mathrm{Im}[\beta_k \Phi_k(z_k) + \gamma_k \Psi_k(\varsigma_k)] \\
u^{(k)} &= 2\,\mathrm{Re}[\rho_k \varphi_k(z_k) + r_k \psi_k(\varsigma_k)] \\
v^{(k)} &= -2\,\mathrm{Im}[\beta_k r_k \varphi_k(z_k) + \gamma_k \rho_k \psi_k(\varsigma_k)]
\end{aligned} \tag{1.6}$$

$$z_k = x + i\beta_k y,\ \ \varsigma_k = x + i\gamma_k y,\ \ \Phi_k(z_k) = \varphi_k'(z_k),\ \ \Psi_k(\varsigma_k) = \psi_k'(z_k),\ \ k = 1,2$$

Here $\pm i\beta_k$, $\pm i\gamma_k$ are the roots of the characteristic equation

$$\mu^4 + \left(\frac{E_k}{G_k} - 2\nu_k \right) \mu^2 + \frac{E_k}{E_k^*} = 0, \quad (\beta_k > \gamma_k),$$

(E_k, E_k^*) are the Young's modulus with respect to the principal (Ox, Oy) direction respectively, G_k are the shear modulus, ν_k are Poisson's ratios of the plane materials, respectively.

The problem with conditions (1.1)-(1.5) reduced to finding the functions $\Phi_k(z_k), \Psi_k(\varsigma_k),\ (k=1,2)$ which are holomorphic in the regions $S^{(k)}$ respectively, and satisfies the following boundary conditions:

$$
\left.\begin{aligned}
&2\,\mathrm{Re}[\Phi_1^+(x)-\Phi_1^-(x)+\Psi_1^+(x)-\Psi_1^-(x)]=p(x)\\
&2\,\mathrm{Im}[\beta_1(\Phi_1^+(x)-\Phi_1^-(x))+\gamma_1(\Psi_1^+(x)-\Psi_1^-(x)]=\tau(x)\\
&\mathrm{Re}[\rho_1(\Phi_1^+(x)-\Phi_1^-(x))+r_1(\Psi_1^+(x)-\Psi_1^-(x))]=0\\
&\mathrm{Im}[\beta_1 r_1(\Phi_1^+(x)-\Phi_1^-(x))+\gamma_1\rho_1(\Psi_1^+(x)-\Psi_1^-(x))]=0
\end{aligned}\right\}\quad 0<x<1 \tag{1.7}
$$

$$
\begin{aligned}
&\mathrm{Re}[\beta_1^2\Phi_1(t_1)+\gamma_1^2\Psi_1(\sigma_1)]=\mathrm{Re}[\beta_2^2\Phi_2(t_2)+\gamma_2^2\Psi_2(\sigma_2)]\\
&\mathrm{Im}[\beta_1\Phi_1(t_1)+\gamma_1\Psi_1(\sigma_1)]=\mathrm{Im}[\beta_2\Phi_2(t_2)+\gamma_2\Psi_2(\sigma_2)]\\
&\mathrm{Im}[\rho_1\beta_1\Phi_1(t_1)+r_1\gamma_1\Psi_1(\sigma_1)]=\mathrm{Im}[\rho_2\beta_2\Phi_2(t_2)+r_2\gamma_2\Psi_2(\sigma_2)]\\
&\mathrm{Re}[\beta_1^2 r_1\Phi_1(t_1)+\gamma_1^2\rho_1\Psi_1(\sigma_1)]=\mathrm{Re}[\beta_2^2 r_2\Phi_2(t_2)+\gamma_2^2\rho_2\Psi_2(\sigma_2)]
\end{aligned} \tag{1.8}
$$

where $t_k=i\beta_k y,\ \sigma_k=i\gamma_k y,\ \rho_k=-\dfrac{\beta_k^2+\nu_k}{E_k},\ r_k=-\dfrac{\gamma_k^2+\nu_k}{E_k},\ k=1,2$

System (1.7)has the unique solution

$$
\begin{aligned}
&\Phi_1^+(x)-\Phi_1^-(x)=\frac{-r_1\beta_1 p(x)+i\rho_1\tau(x)}{2\beta_1(\rho_1-r_1)}\\
&\Psi_1^+(x)-\Psi_1^-(x)=\frac{\rho_1\gamma_1 p(x)-ir_1\tau(x)}{2\gamma_1(\rho_1-r_1)},\qquad 0<x<1
\end{aligned} \tag{1.9}
$$

In view of the fact that $\tau(x)=0,\ p(x)=0$ when $x>1$, the general solution of problem (1. 9) can be represented in the form:

$$
\begin{aligned}
&\Phi_1(z_1)=\frac{ir_1}{4\pi(\rho_1-r_1)}\int_0^1\frac{N_1(t)dt}{t-z_1}+w_1(z_1)\equiv ir_1 w_0(z_1)+w_1(z_1),\\
&\Psi_1(\zeta_1)=-\frac{i\rho_1}{4\pi(\rho_1-r_1)}\int_0^1\frac{N_2(t)dt}{t-\zeta_1}+w_2(\zeta_1)\equiv -i\rho_1 w_0(\varsigma_1)+w_2(\varsigma_1),
\end{aligned} \tag{1.10}
$$

$$
N_1(t)=p(t)-i\frac{\rho_1}{r_1\beta_1}\tau(t),\qquad N_2(t)=p(t)-i\frac{r_1}{\rho_1\gamma_1}\tau(t),
$$

where $w_1(z_1)$ and $w_2(\varsigma_1)$ are unknown analytic functions in the half-planes $\operatorname{Re} z_1 > 0,\ \operatorname{Re}\varsigma_1 > 0$ respectively, which will be defined by using the conditions (1.8).

We will now introduce the boundary values of functions $\Phi_1(z_1)$ and $\Psi_1(\varsigma_1)$, expressed by formulae (1.10), into equalities (1. 8) and multiply expressions obtained by $\frac{1}{2\pi i}\frac{dt}{t-z}$, $t = iy,\ z = x + iy,\ x > 0$ we integrate along the imaginary axis and use the fact that if $\Phi(z)$ is a holomorphic function in the half-plane $\operatorname{Im} z > 0\ (\operatorname{Im} z < 0)$, then $\overline{\Phi(iy)}$ is the boundary value of the function $\overline{\Phi(-\overline{z})}$, holomorphic in the half-plane $\operatorname{Im} z < 0\ (\operatorname{Im} z > 0)$. As a result, using Cauchy's theorem and formula, we obtain the following system

$$\beta_1^2 w_1(\beta_1 z) + \gamma_1^2 w_2(\gamma_1 z) - \beta_2^2\overline{\Phi_2(-\beta_2\overline{z})} - \gamma_2^2\overline{\Psi_2(-\gamma_2\overline{z})} = \\ = -ir_1\beta_1^2\overline{w_0(-\beta_1\overline{z})} + i\rho_1\gamma_1^2\overline{w_0(-\gamma_1\overline{z})}$$

$$\beta_1 w_1(\beta_1 z) + \gamma_1 w_2(\gamma_1 z) + \beta_2\overline{\Phi_2(-\beta_2\overline{z})} + \gamma_2\overline{\Psi_2(-\gamma_2\overline{z})} = \\ = ir_1\beta_1\overline{w_0(-\beta_1\overline{z})} - i\rho_1\overline{\gamma_1 w_0(-\gamma_1\overline{z})}$$

$$\rho_1\beta_1 w_1(\beta_1 z) + r_1\gamma_1 w_2(\gamma_1 z) + \rho_2\beta_2\overline{\Phi_2(-\beta_2\overline{z})} + \gamma_2\overline{r_2\Psi_2(-\gamma_2\overline{z})} = \\ = ir_1\rho_1\beta_1\overline{w_0(-\beta_1\overline{z})} - i\rho_1 r_1\gamma_1\overline{w_0(-\gamma_1\overline{z})}$$

$$\beta_1^2 r_1 w_1(\beta_1 z) + \gamma_1^2\rho_1 w_2(\gamma_1 z) - \beta_2^2 r_2\overline{\Phi_2(-\beta_2\overline{z})} - \gamma_2^2\rho_2\overline{\Psi_2(-\gamma_2\overline{z})} = \\ = -ir_1^2\beta_1^2\overline{w_0(-\beta_1\overline{z})} + i\rho_1^2\gamma_1^2\overline{w_0(-\gamma_1\overline{z})}$$

Solving this system for functions $w_1(\beta_1 z)$ and $w_2(\gamma_1 z)$, and replacing z by z_1/β_1 and ς_1/γ_1 respectively, one obtains

$$w_1(z_1) = \frac{iI_1}{\Delta}\overline{w_0(-\overline{z}_1)} + \frac{iI_2}{\Delta}\overline{w_0(-\frac{\gamma_1}{\beta_1}\overline{z}_1)},$$

$$w_2(\varsigma_1) = \frac{iI_1^*}{\Delta}\overline{w_0(-\frac{\beta_1}{\gamma_1}\overline{\varsigma}_1)} + \frac{iI_2^*}{\Delta}\overline{w_0(-\overline{\varsigma}_1)}, \tag{1.11}$$

For functions $\Phi_2(-\beta_2 z)$ and $\Psi_2(-\gamma_2 z)$ with this notation $-\beta_2 z = z_2, -\gamma_2 z = \varsigma_2$, we have

$$\Phi_2(z_2) = -\frac{iI_3}{\Delta} w_0(\frac{\beta_1}{\beta_2} z_2) - \frac{iI_4}{\Delta} w_0(\frac{\gamma_1}{\beta_2} z_2),$$

$$\Psi_2(\varsigma_2) = -\frac{iI_3^*}{\Delta} w_0(\frac{\beta_1}{\gamma_2} \varsigma_2) - \frac{iI_4^*}{\Delta} w_0(\frac{\gamma_1}{\gamma_2} \varsigma_2),$$

where

$$I_1 = -\Delta_{11} r_1 \beta_1^2 + \Delta_{21} r_1 \beta_1 + \Delta_{31} r_1 \rho_1 \beta_1 - \Delta_{41} \beta_1^2 r_1^2,$$

$$I_2 = \Delta_{11} \rho_1 \gamma_1^2 - \Delta_{21} \rho_1 \gamma_1 - \Delta_{31} \rho_1 r_1 \gamma_1 + \Delta_{41} \rho_1^2 \gamma_1^2,$$

$$I_1^* = -\Delta_{12} r_1 \beta_1^2 + \Delta_{22} r_1 \beta_1 + \Delta_{32} r_1 \rho_1 \beta_1 - \Delta_{42} \beta_1^2 r_1^2,$$

$$I_2^* = \Delta_{12} \rho_1 \gamma_1^2 - \Delta_{22} \rho_1 \gamma_1 - \Delta_{32} \rho_1 r_1 \gamma_1 + \Delta_{42} \rho_1^2 \gamma_1^2,$$

$$I_3 = -\Delta_{13} r_1 \beta_1^2 + \Delta_{23} r_1 \beta_1 + \Delta_{33} r_1 \rho_1 \beta_1 - \Delta_{43} \beta_1^2 r_1^2$$

$$I_4 = \Delta_{13} \rho_1 \gamma_1^2 - \Delta_{23} \rho_1 \gamma_1 - \Delta_{33} \rho_1 r_1 \gamma_1 + \Delta_{43} \rho_1^2 \gamma_1^2$$

$$I_3^* = -\Delta_{14} r_1 \beta_1^2 + \Delta_{24} r_1 \beta_1 + \Delta_{34} r_1 \rho_1 \beta_1 - \Delta_{44} \beta_1^2 r_1^2$$

$$I_4^* = \Delta_{14} \rho_1 \gamma_1^2 - \Delta_{24} \rho_1 \gamma_1 - \Delta_{34} \rho_1 r_1 \gamma_1 + \Delta_{44} \rho_1^2 \gamma_1^2$$

$$\Delta = \begin{vmatrix} \beta_1^2 & \gamma_1^2 & -\beta_2^2 & -\gamma_2^2 \\ \beta_1 & \gamma_1 & \beta_2 & \gamma_2 \\ \rho_1\beta_1 & r_1\gamma_1 & \rho_2\beta_2 & r_2\gamma_2 \\ \beta_1^2 r_1 & \gamma_1^2 \rho_1 & -\beta_2^2 r_2 & -\gamma_2^2 \rho_2 \end{vmatrix},$$

$\Delta_{ij}(i, j = 1,2,3,4)$ are the cofactors of the corresponding matrix elements.

Boundary conditions (1.2) are equivalent to the relations:

$$\frac{1}{E(x)} \int_0^x [\tau_1(t) - \tau_1^0(t)] dt - [\rho_1 \Phi_1(x) + \rho_1 \overline{\Phi_1(x)} + r_1 \Psi_1(x) + r_1 \overline{\Psi_1(x)}] = k_0 \tau'(x)$$

$$\frac{1}{D(x)} \int_0^x dt \int_0^t [p_1^0(\tau) - p_1(\tau)] d\tau -$$

$$-i \frac{d}{dx} \left[\beta_1 r_1 \left(\Phi_1(x) - \overline{\Phi_1(x)} \right) + \gamma_1 \rho_1 \left(\Psi_1(x) - \overline{\Psi_1(x)} \right) \right] = m_0 p_1''(x) \qquad (1.\ 12)$$

Substituting expressions (1.10) and (1.11) into (1.12) we obtain the integro-differential equations on the interval $0 < x < 1$

$$\frac{\psi(x)}{E(x)}-\frac{1}{2\pi}\int_0^1 Q(t,x)\psi'(t)dt-k_0\psi''(x)=f_1(x), \tag{1.13}$$

$$\frac{\varphi(x)}{D(x)}+\frac{1}{2\pi}\frac{d}{dx}\int_0^1 R(t,x)\varphi''(t)dt+m_0\varphi^{IV}(x)=f_2(x), \tag{1.14}$$

$$\psi(1)=0, \quad \varphi(1)=0, \quad \varphi'(1)=0$$

where

$$Q(t,x)=\frac{\lambda_1}{t-x}+\frac{\lambda_2}{t+x}+\frac{\lambda_3}{\beta_1 t+\gamma_1 x}+\frac{\lambda_4}{\gamma_1 t+\beta_1 x}$$

$$R(t,x)=\frac{k_1}{t-x}+\frac{k_2}{t+x}+\frac{k_3}{\beta_1 t+\gamma_1 x}+\frac{k_4}{\gamma_1 t+\beta_1 x}$$

$$\psi(x)=\int_0^x[\tau(t)-\tau_0(t)]dt\,,\ \varphi(x)=\int_0^x dt\int_0^t[p_0(\tau)-p(\tau)]d\tau\,,$$

$$f_1(x)=\frac{1}{2\pi}\int_0^1 Q(t,x)\tau_0(t)dt+k_0\tau_0'(x),$$

$$f_2(x)=m_0p_0''(x)+\frac{1}{2\pi}\frac{d}{dx}\int_0^1 R(t,x)p_0(t)dt$$

$$\lambda_1=\frac{\rho_1^2\gamma_1-r_1^2\beta_1}{(\rho_1-r_1)\beta_1\gamma_1},\ \lambda_2=\frac{\rho_1^2\gamma_1I_1+r_1^2\beta_1I_2^*}{\Delta\beta_1\gamma_1(\rho_1-r_1)},\ \lambda_3=\frac{-I_2\rho_1^2}{\Delta r_1(\rho_1-r_1)},\ \lambda_4=\frac{-I_1^*r_1^2}{\Delta\rho_1(\rho_1-r_1)},$$

$$k_1=\frac{\beta_1r_1^2+\gamma_1\rho_1^2}{\rho_1-r_1},\ \ k_2=\frac{\beta_1r_1I_1+\gamma_1\rho_1I_2^*}{\Delta(\rho_1-r_1)},\ \ k_3=\frac{\beta_1^2r_1I_2}{\Delta(\rho_1-r_1)},\ \ k_4=\frac{\gamma_1^2\rho_1I_1^*}{\Delta(\rho_1-r_1)}.$$

Exact solution of equation (1.14)

Under the condition, when the inclusion-beam is loaded only with normal forces and bending stiffness of the inclusion varies linearly, i.e. $D(x)=d_0x^3$, $m_0(x)=m_0x$,the equation (1.14) and the corresponding boundary conditions take the form

$$\frac{\varphi(x)}{D(x)}-\frac{1}{2\pi}\frac{d}{dx}\int_0^1 R(t,x)\varphi''(t)dt+[m_0(x)\varphi''(x)]''=f_2(x), \qquad 0<x<1 \tag{2.1}$$

$$\varphi(1)=0 \qquad \varphi'(1)=0$$

$$f_2(x) = -\frac{d}{dx}\int_0^1 \frac{p_0(t)dt}{t-x} + m_0(x)p_0''(x),$$

The solution of equation(2.1) is sought in the class of functions [17]

$$\varphi, \varphi', \varphi'', \varphi''' \in H([0,1]),\ \ \varphi^{IV} \in H((0,1)).$$

The change of variables $x = e^{\xi}$, $t = e^{\varsigma}$ in equation (2.1) gives

$$\frac{\varphi_0(\xi)}{d_0 e^{3\xi}} - \frac{1}{2\pi e^{\xi}}\frac{d}{d\xi}\int_{-\infty}^{0} e^{-2\varsigma} R\left(1, e^{\xi-\varsigma}\right)[\varphi_0''(\varsigma) - \varphi_0'(\varsigma)]d\varsigma +$$

$$+m_0 e^{-3\xi}[\varphi_0{}^{iv}(\xi) - 4\varphi_0'''(\xi) + 5\varphi_0''(\xi) - 2\varphi_0'(\xi)] == f_2\left(e^{\xi}\right), \qquad -\infty < x < 0,$$

where $\varphi_0(\xi) = \varphi(e^{\xi})$

Subjecting both part of this equation to generalized Fourier transform [18] we obtain the following Riemann boundary value condition:

$$\Phi^-(s)G(s) = \Psi^+(s) + F(s), \qquad |s| < \infty \tag{2.2}$$

where

$$G(s) = 1 + \frac{d_0}{2}\left[k_1 s\coth \pi s + k_2\frac{s}{\operatorname{sh}\pi s} + k_3\frac{e^{is\mu}}{\gamma_1}\frac{s}{\operatorname{sh}\pi s} + k_4\frac{e^{-is\mu}}{\beta_1}\frac{s}{\operatorname{sh} s\pi s}\right](s-i) +$$

$$+\lambda_2^4(s^4 - 4is^3 - 5s^2 + 2is), \qquad \mu = \ln\frac{\beta_1}{\gamma_1}, \qquad \lambda_2^4 = m_0 d_0$$

$$\Phi^-(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0}\varphi(e^{\varsigma})e^{is\varsigma}d\varsigma, \quad F(s) = \sqrt{\frac{2}{\pi}}\int_{-\infty}^{0} e^{3\xi} f_2(e^{\xi})e^{is\xi}d\xi,$$

$$\Psi^+(s) = \frac{1}{\sqrt{2\pi}}\int_0^{\infty}\psi(\xi)e^{is\xi}d\xi$$

$$\psi(y) = \begin{cases} 0, & y < 0 \\ e^{2y}\dfrac{d}{dy}\displaystyle\int_{-\infty}^{\infty} e^{-2s}R(1, e^{y-s})[\varphi_0''(s) - \varphi_0'(s)]ds, & y > 0 \end{cases}$$

The condition (2.2) can be represented as

$$\Phi^-(s)(1+\sqrt{i}\lambda_2 s)(1+i\sqrt{i}\lambda_2 s)G_0(s) = \frac{\Psi^+(s)}{(1-\sqrt{i}\lambda_2 s)(1-i\sqrt{i}\lambda_2 s)} + H(s) \tag{2. 3}$$

where

$$G(s) = \frac{G(s)}{1+\lambda_2^4 s^4}(1+i\lambda_2^2 s^2)(1-i\lambda_2^2 s^2) \equiv$$

$$\equiv G_0(s)(1-\sqrt{i}\lambda_2 s)(1-i\sqrt{i}\lambda_2 s)(1+\sqrt{i}\lambda_2 s)(1+i\sqrt{i}\lambda_2 s)$$

$$H(s) = \frac{F(s)}{(1-\sqrt{i}\lambda_2 s)(1-i\sqrt{i}\lambda_2 s)}, \ G_0(s) = \frac{G(s)}{1+\lambda_2^4 s^4}.$$

By virtue of functions $\Phi^-(s)$ and $\Psi^+(s)$ definition, they will be boundary values of the functions which are holomorphic in the upper and lower half-planes, respectively.

The problem can be formulated as follows: it is required to determine the function $\Psi^+(z)$, holomorphic in the half-plane $\operatorname{Im} z > 0$ and which vanishes at infinity, and the function $\Phi^-(z)$, holomorphic in the half-plane $\operatorname{Im} z < 1$, (with the exception of a finite number of zeros of function $G(z)$) which vanishes at infinity and are continuous on the real axis by condition (2.3).

Since $\operatorname{Re} G_0(s) > 0$ and $G_0(\infty) = G_0(-\infty) = 1$, we have $\operatorname{Ind} G_0(s) = 0$.

The solution of this problem has the form [17]

$$\Phi^-(z) = \frac{\tilde{X}(z)}{(1+\sqrt{i}\lambda_2 s)(1+i\sqrt{i}\lambda_2 s)}, \qquad \operatorname{Im} z \le 0;$$

$$\Psi^+(z) = \tilde{X}(z)(1-\sqrt{i}\lambda_2 s)(1-i\sqrt{i}\lambda_2 s), \quad \operatorname{Im} z > 0$$

$$\Phi^-(z) = (\Psi^+(z) + F(z))G^{-1}(z), \ 0 < \operatorname{Im} z < 1 \tag{2.4}$$

where

$$\tilde{X}(z) = X(z)\left\{-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{H(t)dt}{X^+(t)(t-z)}\right\}, \qquad X(z) = \exp\left\{\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\ln G_0(t)dt}{t-z}\right\}.$$

(here the integral should be understood in the sense of the Cauchy principal value).

Using the formula $\varphi''(x) = \dfrac{\varphi_0''(\ln x) - \varphi_0'(\ln x)}{x^2}$ and applying the inverse Fourier transformation

$$\varphi_0'(\ln x) = -\frac{i}{\sqrt{2\pi}}\int_{-\infty}^{\infty} s\Phi^-(s)e^{-is\ln x}ds, \ \varphi_0''(\ln x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} s^2\Phi^-(s)e^{-is\ln x}ds$$

we will investigate the behavior of the function $p_0(x) - p(x) = \psi''(x)$ in the neighborhood of the points $z = 0$ and $z = 1$.

We obtain by an inverse transformation: $p_0(x) - p(x) = O(1), \quad x \to 1-$.

The poles of the function $\Phi^-(z)$ in the domain $D_0 = \{z : 0 < \operatorname{Im} z < 1\}$ may be zeros of the function $G(z)$. It can be shown that the function $G(z)$ has no zeros in the strip $0 < \operatorname{Im} z < 2$. Then, applying Cauchy's theorem to the functions $e^{-i\xi z} iz\Phi^-(z)$, $e^{-i\xi z} z^2 \Phi^-(z)$ we obtain the following estimate

$$p_0(x) - p(x) = O(x^{y_0 - 2 + ix_0}), \qquad x \to 0+, \qquad y_0 > 2 \tag{2.5}$$

where $z_0 = x_0 + iy_0$ is zero of the function $G(z)$ with a minimal imaginary part and with $x_0 \neq 0$, consequently we have oscillating stress singularities.

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