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A HYDRODYNAMIC THRUST BEARING LUBRICATED BY A NON-NEWTONIAN GIESEKUS FLUID

Ilya I. Kudish, Sergei S. Volkov, Andrey S. Vasiliev

Keywords: fluid with non-Newtonian nonlinear Giesekus model, hydrodynamic lubrication problem for a thrust bearing, perturbation analysis, analytical approximate solution

И.И.Кудиш, С.С.Волков, А.С.Васильев

Гидродинамический упорный подшипник, смазываемый неньютоновской жидкостью Гизекуса

Ключевые слова: жидкость с неньютоновской нелинейной моделью Гизекуса, гидродинамическая задача смазки упорного подшипника, анализ методом возмущений, аналитическое приближенное решение

Существует огромный объем исследований гидродинамических и упругогидродинамических задач смазки для смазок с ньютоновской реологией. Смазочные материалы с ньютоновской реологией не проявляют обычно наблюдаемого экспериментального поведения относительно высокой вязкости при низких напряжениях и относительно низкой вязкости при высоких напряжениях. В этой статье мы расширяем ранее проведенный анализ смазочных материалов с неньютоновским поведением Гизекуса для случая моделирования упорного подшипника. Основная цель статьи – получить аналитическое решение для упорного гидродинамического подшипника, смазываемого жидкостью, с реологией Гизекуса. Эта цель достигается тщательным применением методов возмущений. Получено трехчленное приближенное аналитическое решение и проанализирована его зависимость от входных параметров задачи.

Ի.Ի.Կուդիշ, Ս.Ս.Վոլկով, Ա.Ս.Վասիլև

Գիզեկուսի ոչ նյուտոնյան հեղուկով յուղվող հիդրոդինամիկական հենակային առանցքակալ

Հիմնաբառեր՝ Գիզեկուսի ոչ նյուտոնյան ոչ գծային մոդելով հեղուկ, հենակային առանցքակալի յուղման հիդրոդինամիկական նդիր, վերլուծություն գրգռումների եղանակով, մոտավոր անալիտիկ լուծում:

Նյուտոնյան ռեոլոգիայով քսանյութերով հիդրոդինամիկական և առաձգահիդրոդինամիկական յուղման խնդիրների վերաբերյալ գոյություն ունի հետազոտությունների մի հսկա ծավալ: Նյուտոնյան ռեոլոգիայով քսանյութերը չեն ցուցադրում սովորաբար նկատվող փորձարարական վարքագիծ՝ ցածր լարումների դեպքում համեմատաբար բարձր մածուցիկություն և բարձր լարումների դեպքում համեմատաբար ցածր մածուցիկություն: Այս հոդվածում մենք ընդլայնում ենք Գիզեկուսի ոչ նյուտոնյան վարքագծով քսանյութերի մեր նախորդ վերլուծությունը հենակային առանցքակալների մոդելավորման համար: Հոդվածի հիմնական նպատակն է Գիզեկուսի ռեոլոգիայով օժտված հեղուկով յուղված հենակային հիդրոդինամիկական առանցքակալի համար ստանալ անալիտիկ լուծում: Այս նպատակը հասանելի է դառնում գրգռման մեթոդների մանրազննին կիրառմամբ: Ստացված է եռանդամային մոտավոր անալիտիկ լուծում և վերլուծված է դրա կախվածությունը խնդրի մուտքային պարամետրերից:

There exists a huge volume of studies of hydrodynamic and elastohydrodynamic lubrication problems for lubricants with Newtonian rheology. Lubricants with Newtonian rheology do not exhibit the usually observed experimentally behavior of having relatively high viscosity for low stresses and relatively low viscosity for high stresses. In this paper we extend the earlier conducted analysis of lubricants with a non-Newtonian Giesekus behavior for the case of thrust bearing modeling. The main goal of the paper is to obtain an analytical solution for a hydrodynamically thrust bearing lubricated by a fluid with the Giesekus rheology. This goal is achieved by careful application of perturbation methods. A three-term approximate analytical solution is obtained and its dependence on the problem input parameters is analyzed.

Introduction

Over the years the modern automotive industry as well as various bearing and gear setups demand more and more efficient lubrication to reduce friction losses, contact energy losses, and to increase joint fatigue durability. Obviously, even a small increase in lubricated joint efficiency multiplied by millions of cars and lots of other moving mechanisms can be quite significant in reducing emissions, fuel, and material required for joint manufacturing worldwide. Frictional losses are associated with a number of specific components among which are engines, bearings, and gears. Therefore, understanding tribological characteristics of lubricated contacts may help in reducing frictional losses, increasing fluid economy and fatigue durability. There is a large number of papers dedicated to studying hydrodynamic and elastohydrodynamic lubrication contacts with Newtonian lubricants [1] - [25]. These paper cover problems under isothermal and thermal conditions for smooth and textured surfaces etc.

Several decades ago lubrication industry started using formulated lubricants represented by a base stock oils (described by Newtonian rheology) with some polymeric additives. These additives make the rheology of formulated lubricants non-Newtonian. Most of the existing and usually used non-Newtonian lubricant rheologies [26] are linear rheological fluid models such as Maxwell, Jeffrey, various Oldroyd-B models, etc. A review of such models is given in [41]. These models are designed to introduce into consideration an important fluid parameter such as its relaxation time related to the structure of the polymeric additive. Some studies of these kind of lubricating fluids can be found in [27] - [32]. Various elastohydrodynamic and hydrodynamic problems for lubricants with generalized Newtonian rheology were considered in [33, 34]. Some other elastohydrodynamic lubrication problems for functionally graded materials and hydrodynamic problems for solids without coatings, with a single and double coatings and Newtonian lubricants were considered in [35] - [37] and [38] - [40], respectively.

The main defect of these kind of models is their inability to adequately describe fluid rheological behavior for low and high fluid stresses when usually lubricant viscosity approaches to two different limiting values. The rheological fluid model that is free of the just mentioned defect is the Giesekus model [26]. Specifically, besides introducing the fluid relaxation time this model provides for relatively high fluid viscosity at low fluid stresses and relatively low fluid viscosity at relatively high fluid stresses. This model is nonlinear and, therefore, it is much harder to analyze lubrication problems involving lubricants with such a rheology. For a relatively simple case of a Giesekus fluid flow between two parallel flat surfaces is considered in [42, 43].

There is a paper on lubrication of a two-dimensional model of a thrust bearing with a fluid with the Giesekus rheological model [44]. The paper analysis is performed with the help of a perturbation method. However, the paper contains a number of shortcomings. For example, all convective terms in the equations of fluid motion and fluid rheological equations are omitted, Reynolds equations solved are incomplete, the perturbation analysis performed is not consistent in the case when the thrust bearing is of the same order as the Giesekus fluid mobility parameter α, etc. One limiting case of hydrodynamic lubricated contact for the case of two moving rigid cylinders separated by a thin layer of an incompressible lubricant with the Giesekus rheology is considered in [45].

In this paper, the Giesekus rheology is used for modeling friction between one rigid surfaces moving over another rigid surface at rest. The surfaces are separated by a incompressible fluid described by the Giesekus rheology. The problem is analyzed using the regular perturbation method. The approximate solution is obtained in an analytical form. Many applications of perturbation techniques to steady problems can be founded in [46]. Also, it can be applied to dynamic problems, for example see [47].

The paper is organized as following. In the first section, the formulation of the hydrodynamic lubrication problem for a line contact is presented. In the second section, the proper simplification of the rheology equations and the equations of the motion pertinent to the case of steady lubricant flow in a narrow long channel is described. The third and fourth sections are dedicated to obtaining the components of lubricant velocity and derivation of Reynolds equations of different order and their analytical solutions, respectively. Some specific examples of the obtained solution and their analysis are presented in the last section. In particular, some examples of pressure distributions, energy loss etc. are provided.

1 Formulation of the Lubrication Problem

Let us consider a steady plane problem for a lubricated contact modeling a two-dimensional hydrodynamic thrust bearing (see Fig. 1) lubricated by an incompressible non-Newtonian Giesekus fluid [26] with constant viscosity μ and relaxation time λ_1. The coordinate system is introduced in such a way that the $x-$axis is directed along the surface of the rigid runner moving with the linear velocity $\overline{u_1}$ while the $z-$axis is perpendicular to it and directed upward. The $y-$axis is directed in the solids. The fixed rigid pad (the linear velocity of which is $\overline{u_2} = 0$) and the runner are completely separated by the lubrication film. The components of the lubricant velocity are represented by functions $u(x, y, z)$, $v(x, y, z) = 0$, and $w(x, y, z)$. The problem parameters are independent of the coordinate y. The equations of the motion of such a fluid are described by the solvent and additive rheology equations as follows [33, 34]

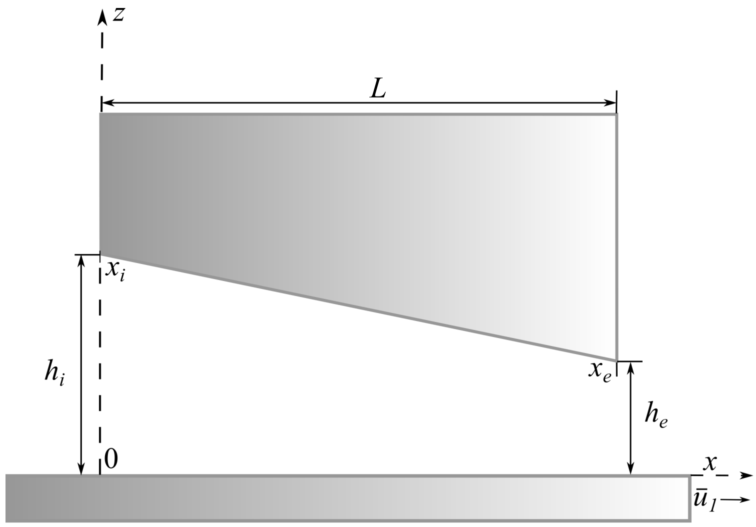

Fig. 1: The general view of a lubricated contact.

$$u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = \frac{1}{\rho}(\frac{\partial p_{xx}}{\partial x} + \frac{\partial p_{zx}}{\partial z}),$$

$$\frac{\partial p_{xy}}{\partial x} + \frac{\partial p_{zy}}{\partial z} = 0, \tag{1.1}$$

$$u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} = \frac{1}{\rho}(\frac{\partial p_{zx}}{\partial x} + \frac{\partial p_{zz}}{\partial z}).$$

In addition to that for an incompressible fluid with the fluid density $\rho(x, z) = constant$ we have the continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0. \tag{1.2}$$

In this case the stress tensor components are as follows

$$p_{xx} = -p + \tau_{xx},\ p_{xy} = \tau_{xy} = 0,\ \ p_{zx} = \tau_{zx},$$
$$p_{zz} = -p + \tau_{zz},\ p_{zy} = \tau_{zy} = 0, \tag{1.3}$$

where p is the pressure and $\tau_{xx},\ \tau_{xy},\ \tau_{zx},\ \tau_{zz}$, and τ_{zy} are additional stress components acting in the corresponding directions. These tensor components satisfy the Giesekus fluid model which is a nonlinear model and takes into account the degree to which the additive polymeric molecules are aligned with the lubricant flow which is characterized by the mobility parameter $\alpha,\ 0 \leq \alpha \leq 1$. The rheological equations are as follows [26]

$$\begin{gathered}\tau = \tau_s + \tau_p,\ \mu = \mu_s + \mu_p,\\ \tau_s = \mu_s \dot{\gamma},\\ \tau_p + \lambda_1 \tau_{p(1)} - \alpha \tfrac{\lambda_1}{\mu_p}\{\tau_p \cdot \tau_p\} = \mu_p \dot{\gamma},\end{gathered} \tag{1.4}$$

where τ is the full stress tensor while τ_s and τ_p are the solvent and polymer stress tensors, respectively, μ_s and μ_p are the constant solvent and polymer dynamic viscosities, $\dot{\gamma}$ is the deformation tensor [26], and λ_1 is the constant relaxation time.

In (1.4) we used the the definitions of the tensor operators $\tau_{p(1)}$ and $\{\tau_p \cdot \tau_p\}$ from [26].

Assuming that we have no slip and no penetration conditions on the solid surfaces for u and w we have the following boundary conditions on the lubricated surfaces

$$u(x,0) = \overline{u_1},\ u(x,h(x)) = 0, \tag{1.5}$$

$$w(x,0) = w(x,h(x)) = 0. \tag{1.6}$$

The gap between the runner and pad is described by the function

$$h(x) = h_i + mx,\ m = \tfrac{h_e - h_i}{L} < 0,\ m = m_0\epsilon,\ m_0 = O(1),\ \epsilon \ll 1, \tag{1.7}$$

where L is the actual length of the bearing, h_i and h_e are the gaps between the runner and pad at the inlet and exit from the contact, respectively, and $\epsilon = h_i/L \ll 1$ is a small parameter of the problem.

As outside of the bearing the lubricant pressure is atmospheric which is much lower the pressure in the lubricated contact for pressure p we have the following boundary conditions

$$p(0,z) = p(L,z) = 0. \tag{1.8}$$

It is assumed here that the inlet point in the lubricated contact is located at $x = 0$ while the exit point from the lubricated contact is located at $x = L$.

Our goal is to determine such components of the solution as contact pressure $p(x,z)$, the components of the tensor $\tau(x,z)$ in the fluid and its velocity components $u(x,z)$ and $w(x,z)$. We will find a three-term perturbation solution of the above determined problem in the case when $\epsilon \ll 1$. We will assume that

$$\begin{gathered}\alpha = \alpha_0\epsilon,\ \alpha_0 = O(1),\ \epsilon = \tfrac{h_i}{L} \ll 1,\\ \lambda_1 = \lambda\epsilon,\ \lambda = O(1),\ \epsilon \ll 1.\end{gathered} \tag{1.9}$$

Here α_0 and λ are nonnegative constants. Also, we will assume that

$$Re_0 = \tfrac{\rho \overline{u_1} h_i}{\mu_*} = O(1),\ \alpha \ll 1, \tag{1.10}$$

were Re_0 is the effective local Reynolds number in the lubricant flow and μ_* is the ambient lubricant viscosity.

2 Asymptotic Analysis of the Rheological and Motion Equations

Let us introduce the following dimensionless variables

$$\lambda_1' = \frac{\overline{u_1}}{L}\lambda_1,\ x' = \frac{x}{L},\ \{z', h_i', h_e'\} = \frac{1}{h_i}\{z, h_i, h_e\},\ p' = p\frac{h_i^2}{\mu_* \overline{u_1} L},$$
$$u' = \frac{u}{\overline{u_1}},\ w' = \frac{w}{U_z},\ \{\mu', \mu_s', \mu_p'\} = \frac{1}{\mu_*}\{\mu, \mu_s, \mu_p\},$$
$$\{\tau_{xx}', \tau_{sxx}', \tau_{pxx}', \tau_{zx}', \tau_{szx}', \tau_{pzx}', \tau_{zz}', \tau_{szz}', \tau_{pzz}'\}$$
$$= \frac{h_i}{\mu_* \overline{u_1}}\{\tau_{xx}, \tau_{sxx}, \tau_{pxx}, \tau_{zx}, \tau_{szx}, \tau_{pzx}, \tau_{zz}, \tau_{szz}, \tau_{pzz}\}, \quad (2.1)$$

where U_z is the characteristic velocity of the lubricating fluid in the direction of the $z-$axis.

For simplicity in the further analysis the primes at the dimensionless variables are dropped. Then the dimensionless $h_i = 1$ and the problem solution is searched within the interval $0 \leq x \leq 1$.

Due to nonlinearity and complexity of the problem it is impossible to develop any analytical solutions except the perturbation ones which will be used in this analysis. We will search the problem solution in the form of the following series in ϵ

$$\{\tau_{sxx}(x,z), \tau_{sxz}(x,z), \tau_{szz}(x,z)\} = \{\tau_{sxx0}(x,z), \tau_{sxz0}(x,z),$$
$$\tau_{szz0}(x,z)\} + \epsilon\{\tau_{sxx1}(x,z), \tau_{sxz1}(x,z), \tau_{szz1}(x,z)\} \quad (2.2)$$
$$+\epsilon^2\{\tau_{sxx2}(x,z), \tau_{sxz2}(x,z), \tau_{szz2}(x,z)\} + \ldots,$$

$$\{\tau_{pxx}(x,z), \tau_{pxz}(x,z), \tau_{pzz}(x,z)\} = \{\tau_{pxx0}(x,z), \tau_{pxz0}(x,z),$$
$$\tau_{pzz0}(x,z)\} + \epsilon\{\tau_{pxx1}(x,z), \tau_{pxz1}(x,z), \tau_{pzz1}(x,z)\} \quad (2.3)$$
$$+\epsilon^2\{\tau_{pxx2}(x,z), \tau_{pxz2}(x,z), \tau_{pzz2}(x,z) \ + \ldots,$$

$$p(x,z) = p_0(x,z) + \epsilon p_1(x,z) + \epsilon^2 p_2(x,z) + \ldots,$$
$$u(x,z) = u_0(x,z) + \epsilon u_1(x,z) + \epsilon^2 u_2(x,z) + \ldots, \quad (2.4)$$
$$w(x,z) = w_0(x,z) + \epsilon w_1(x,z) + \epsilon^2 w_2(x,z) + \ldots,$$

where $p_0(x,z)$, $u_0(x,z)$, $w_0(x,z)$, $\tau_{sxx0}(x,z)$, $\tau_{sxz0}(x,z)$, $\tau_{szz0}(x,z)$, $\tau_{pxx0}(x,z)$, $\tau_{pxz0}(x,z)$, $\tau_{pzz0}(x,z)$, $p_1(x,z)$, $u_1(x,z)$, $w_1(x,z)$, $\tau_{sxx1}(x,z)$, $\tau_{sxz1}(x,z)$, $\tau_{szz1}\ (x,z)$, $\tau_{pxx1}(x,z)$, $\tau_{pxz1}(x,z)$, $\tau_{pzz1}(x,z)$, $p_2(x,z)$, $u_2(x,z)$, $w_2(x,z)$, $\tau_{sxx2}(x,z)$, $\tau_{sxz2}(x,z)$, $\tau_{szz2}(x,z)$, $\tau_{pxx2}(x,z)$, $\tau_{pxz2}(x,z)$, and $\tau_{pzz2}(x,z)$ are the unknown main, first-, and second-order approximations of the corresponding functions while the gap $h(x)$ between the runner and pad and functions $h_0(x_0)$ and $h_1(x_0)$ are described by the equations (see (1.7))

$$h(x) = 1 + mx = h_0(x) + \epsilon h_1(x),$$
$$h_0(x) = 1, \; h_1(x) = m_0 x. \tag{2.5}$$

As it will be shown below functions $p_0(x,z)$ and $p_1(x,z)$ are independent of z (i.e. $p_0(x,z) = p_0(x)$ and $p_1(x,z) = p_1(x)$) while functions $p_k(x,z)$ for $k \geq 2$ may depend on both x and z.

It is important to realize that the boundary conditions on the terms of the expansions of $u, \; w$, and p in $\epsilon \ll 1$ are

$$u_0(x,0) = 1, \; u_0(x,1) = 0, \; u_1(x,0) = 0, \; u_1(x,1) = -h_1 \frac{\partial u_0(x,1)}{\partial z},$$
$$u_2(x,0) = 0, \; u_2(x,1) = -h_1 \frac{\partial u_1(x,1)}{\partial z} - \frac{h_1^2}{2} \frac{\partial^2 u_0(x,1)}{\partial z^2}, \tag{2.6}$$

$$w_0(x,0) = 0, \; w_0(x,1) = 0, \; w_1(x,0) = 0, \; w_1(x,1) = -h_1 \frac{\partial w_0(x,1)}{\partial z},$$
$$w_2(x,0) = 0, \; w_2(x,1) = -h_1 \frac{\partial w_1(x,1)}{\partial z} - \frac{h_1^2}{2} \frac{\partial^2 w_0(x,1)}{\partial z^2}. \tag{2.7}$$

Also, from (1.8) we have

$$p_0(0) = p_1(0) = p_2(0) = 0, \; p_0(1) = p_1(1) = p_2(1) = 0. \tag{2.8}$$

Using expansions (2.2)-(2.5) in equations (1.1)-(1.6), and (1.8) taking into account (1.9) and (1.10), and equating terms of the same order of ϵ these equations can be simplified and reduced to solution of the problems for Reynolds equations of order zero

$$\frac{d}{dx} \int\limits_0^1 u_0(x,z) dz = 0, \tag{2.9}$$

of order one

$$\frac{d}{dx} \int\limits_0^{h_0} u_1(x,z) dz = 0, \tag{2.10}$$

and of order two

$$\frac{d}{dx} \{ \int\limits_0^{h_0} u_2(x,z) dz + \frac{h_1^2}{2} (\frac{1}{h_0} - \frac{h_0}{2\mu} \frac{dp_0}{dx}) \} = 0, \tag{2.11}$$

where

$$u_0(x,z) = 1 - z + (z^2 - z) \frac{1}{2\mu} \frac{dp_0}{dx}. \tag{2.12}$$

Keeping in mind that $h_0(x) = 1$ the above mentioned problem for the Reynolds equation of order zero has the form

$$\frac{d}{dx} \{ \frac{1}{6\mu} \frac{dp_0}{dx} - h_0 \} = 0, \; h_0(x) = 1, \; p_0(0) = p_0(1) = 0, \tag{2.13}$$

and its solution is

$$p_0(x) = 0. \tag{2.14}$$

It can be shown that $w_0(x, z) = 0$.

Taking that into account we have

$$\begin{aligned} u_1(x, z) &= (z^2 - z)\tfrac{1}{2\mu}\tfrac{dp_1}{dx} + zh_1, \\ w_1(x, z) &= z^2(\tfrac{1}{2} - \tfrac{z}{3})\tfrac{1}{2\mu}\tfrac{d^2p_1}{dx^2} - \tfrac{z^2}{2}\tfrac{dh_1}{dx}. \end{aligned} \tag{2.15}$$

Here we took into account that $h_0 = 1$, $\frac{dh_0}{dx} = \frac{d^2h_1}{dx^2} = w_0 = p_0 = \frac{dp_0}{dx} = \frac{\partial u_0}{\partial x} = 0$.

Then using (2.11), (2.15), and (2.8) we obtain the problem for the Reynolds equation of order one the solution for which is

$$p_1(x) = 3\mu m_0(x^2 - x). \tag{2.16}$$

Obviously, $u_0(x, z)$ from (2.15) and $p_0(x)$ from (2.14) is as a solution of a Couette fluid flow problem through a channel with a constant cross section with zero pressure gradient. Due to the fact that $p_0(x) = 0$ and $p_1(x)$ takes into account only the varying channel cross section via function $h_1(x)$ (the pad slope m_0) and fluid viscosity μ pressure $p_2(x)$ simultaneously takes into account the fluid viscosity, relaxation time, and polymer mobility factor, i.e. parameters μ, λ, and α_0. In other words, the nonlinear non-Newtonian rheology gets incorporated in the problem solution only on the order level of ϵ^2.

The equations for the second order term $p_2(x, z)$ from the rheology and motion equations have the form

$$\begin{aligned} Re_0\{u_0\tfrac{\partial u_1}{\partial x} + u_1\tfrac{\partial u_0}{\partial x} + w_0\tfrac{\partial u_1}{\partial z} + w_1\tfrac{\partial u_0}{\partial z}\} &= -\tfrac{\partial p_2}{\partial x} + \tfrac{\partial \tau_{xx1}}{\partial x} + \tfrac{\partial \tau_{zx2}}{\partial z}, \\ -\tfrac{\partial p_2}{\partial z} + \tfrac{\partial \tau_{zz1}}{\partial z} + \tfrac{\partial \tau_{zx0}}{\partial x} &= 0. \end{aligned} \tag{2.17}$$

Integrating the second equation in (2.17) we get

$$p_2(x, z) = \tau_{zz1} + \mu\tfrac{\partial u_0}{\partial x} + P_2(x), \tag{2.18}$$

where $P_2(x)$ is an arbitrary function of x. Taking into account solutions (2.14), (2.16) and the fact that $h_0(x) = 1$ the expression for $u_2(x, z)$ and the Reynolds equation of order two (2.11) can be significantly simplified and the problem for $P_2(x)$ can be presented in the form

$$\begin{aligned} &\tfrac{d}{dx}\{\tfrac{dP_2}{dx} + 3h_1\tfrac{dp_1}{dx} - \tfrac{Re_0}{120}(\tfrac{11}{\mu}\tfrac{d^2p_1}{dx^2} - 42\tfrac{dh_1}{dx}) - \tfrac{\mu_p\lambda}{\mu}[9\lambda\alpha_0\tfrac{dp_1}{dx} \\ &-\mu\tfrac{dh_1}{dx} - \tfrac{1}{2}\tfrac{d^2p_1}{dx^2}]\} = 0, \ P_2(0) = P_2(1) = -\mu_p\alpha_0\lambda. \end{aligned} \tag{2.19}$$

The solution to this problem is

$$P_2(x) = 3m_0\{9\mu_p\lambda^2\alpha_0 - \mu m_0(2x + \tfrac{1}{2})\}(x^2 - x) - \mu_p\lambda\alpha_0. \tag{2.20}$$

Using (2.18) and (2.20) we obtain

$$p_2(x,z) = 3\mu m_0\{-m_0(2x+\tfrac{1}{2}) + \tfrac{9\mu_p\lambda^2\alpha_0}{\mu}\}(x^2-x). \tag{2.21}$$

After that we can easily calculate the tensor components in the form

$$\tau_{xx} = \tau_{sxx} + \tau_{pxx},\ \tau_{zx} = \tau_{szx} + \tau_{pzx},\ \tau_{zz} = \tau_{szz} + \tau_{pzz},$$

$$\tau_{sxx0} = \tau_{sxx1} = 0,\ \tau_{sxx2} = 2\mu_s\tfrac{\partial u_1}{\partial x},\ \tau_{szx0} = \mu_s\tfrac{\partial u_0}{\partial z},\ \tau_{szx1} = \mu_s\tfrac{\partial u_1}{\partial z}, \tag{2.22}$$

$$\tau_{szx2} = \mu_s[\tfrac{\partial u_2}{\partial z} + \tfrac{\partial w_0}{\partial x}],\ \tau_{szz0} = \tau_{szz1} = 0,\ \tau_{szz2} = -2\mu_s\tfrac{\partial u_1}{\partial x},$$

$$\tau_{pxx0} = 2\mu_p\lambda(\tfrac{\partial u_0}{\partial z})^2,\ \tau_{pzx0} = \mu_p\tfrac{\partial u_0}{\partial z},\ \tau_{pzz0} = 0,$$

$$\tau_{pxx1} = \mu_p\{10\lambda^3\alpha_0(\tfrac{\partial u_0}{\partial z})^4 + 4\lambda\tfrac{\partial u_0}{\partial z}\tfrac{\partial u_1}{\partial z} + \lambda\alpha_0(\tfrac{\partial u_0}{\partial z})^2 \tag{2.23}$$

$$\tau_{pzx1} = \mu_p\{3\lambda^2\alpha_0(\tfrac{\partial u_0}{\partial z})^3 + \tfrac{\partial u_1}{\partial z}\},\ \tau_{pzz1} = \mu_p\lambda\alpha_0(\tfrac{\partial u_0}{\partial z})^2,$$

$$\tau_{pxx2} = 2\mu_p\{-3\lambda^2\tfrac{\partial^3 u_0}{\partial z^3}w_1 + 42\lambda^5\alpha_0^2(\tfrac{\partial u_0}{\partial z})^6 + 7\lambda^3\alpha_0(\tfrac{\partial u_0}{\partial z})^4$$

$$+20\lambda^3(\tfrac{\partial u_0}{\partial z})^3\alpha_0\tfrac{\partial u_1}{\partial z} + \lambda\tfrac{\partial u_0}{\partial z}(\alpha_0\tfrac{\partial u_1}{\partial z} + 2\tfrac{\partial u_2}{\partial z}) + \lambda(\tfrac{\partial u_1}{\partial z})^2 + \tfrac{\partial u_1}{\partial x}\},$$

$$\tau_{pzx2} = -\mu_p\lambda\tfrac{\partial^2 u_0}{\partial z^2}w_1 - \mu_p\lambda\{u_0\tfrac{\partial^2 u_1}{\partial x\partial z} - 22\lambda^3\alpha_0^2(\tfrac{\partial u_0}{\partial z})^5\} \tag{2.24}$$

$$+\mu_p\lambda^2\alpha_0\{2\alpha_0(\tfrac{\partial u_0}{\partial z})^3 + 9(\tfrac{\partial u_0}{\partial z})^2\tfrac{\partial u_1}{\partial z}\} + \mu_p\{-2\lambda\tfrac{\partial u_0}{\partial z}\tfrac{\partial u_1}{\partial x} + \tfrac{\partial u_2}{\partial z}\},$$

$$\tau_{pzz2} = \mu_p\{6\lambda^3\alpha_0^2(\tfrac{\partial u_0}{\partial z})^4 + 2\lambda\alpha_0\tfrac{\partial u_1}{\partial z}\tfrac{\partial u_0}{\partial z} - 2\tfrac{\partial u_1}{\partial x}\}.$$

After that we can easily calculate the additional pressure created in the contact

$$N_1(x,z) = \tau_{xx}(x,z) - \tau_{zz}(x,z). \tag{2.25}$$

The above expressions are simplified using the fact that $w_0 = \frac{\partial u_0}{\partial x} = \frac{\partial^2 u_0}{\partial z^2} = 0$.

3 Examples of Some Specific Lubrication Problem Solutions and Discussion

Now, let us consider some results which can be extracted from the obtained approximate solution. We will assume that always $\mu = 1$. We will take as the basic set the following values: $\epsilon = 0.05$, $Re_0 = 5$, $\mu_p = 0.25$, $\lambda = 1$, $\alpha_0 = 2$, and $m_0 = -0.5$. The pressure distributions $p(x)$ versus parameters μ_p, λ, α_0, and m_0 while in each case the rest of the parameters are fixed and equal to their basic values are presented in Fig. 2. It is clear that in each case the pressure distributions $p(x)$ are very close to a parabola. That can also be seen from the ratios $p_{rel}(x)$ of pressure $p(x)$ divided by the pressure for a lubricant with the Newtonian rheology coinciding

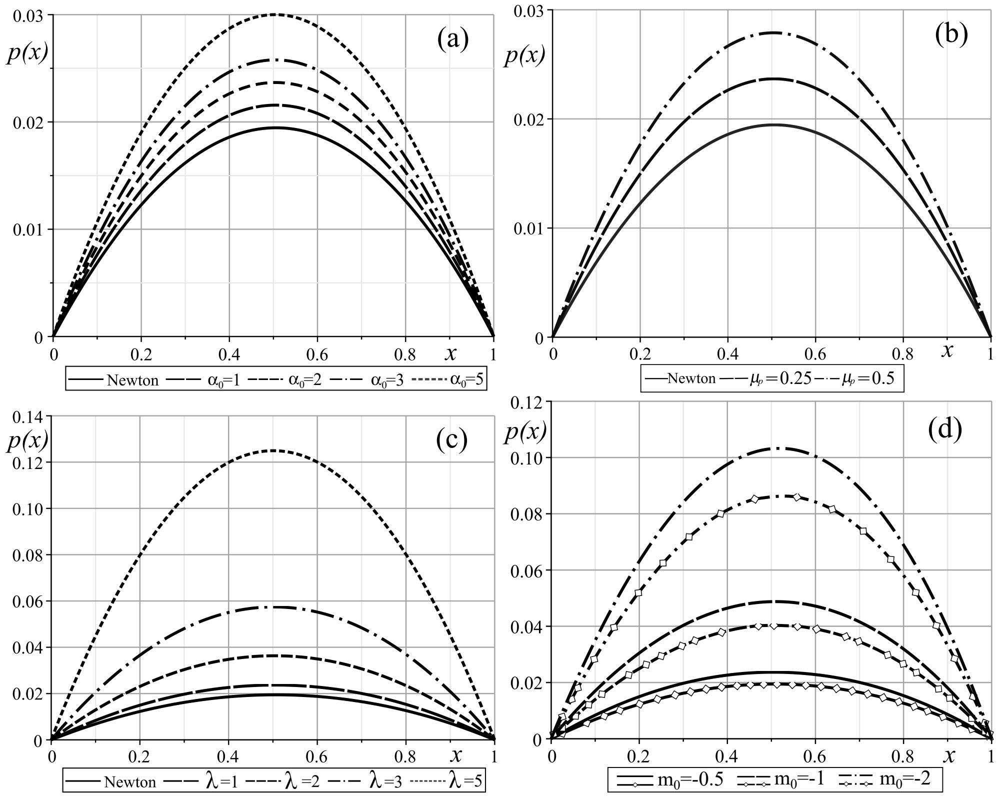

Fig. 2: Curves of the dimensionless pressure $p(x)$ as functions of x for different values of μ_p, λ, α_0, and m_0 as shown in figure legends. In Fig. 2d curves without circles correspond to the case of the dimensionless pressure $p_N(x)$ for a Newtonian fluid ($\mu_p = \lambda = \alpha_0 = 0$). The calculations were made for the following basic set of parameters: $\epsilon = 0.05,\ \mu_p = 0.25,\ \lambda = 1,\ \alpha_0 = 2$, and $mu_0 = -0.5$.

with $p_N(x) = 3\mu m_0 \epsilon\{1 - \epsilon m_0(2x + \frac{1}{2})\}(x^2 - x)$

$$p_{rel}(x) = \frac{1+\epsilon[-m_0(2x+\frac{1}{2})+\frac{9\mu_p\lambda^2\alpha_0}{\mu}]}{1-\epsilon m_0(2x+\frac{1}{2})} + \ldots = 1 + O(\epsilon),\ \epsilon \ll 1. \tag{3.1}$$

which is obtained for $\mu_p = \lambda = \alpha_0 = 0$. Moreover, for $\mu_p > 0$, $\lambda > 0$, and $\alpha_0 > 0$ the contact pressure $p(x)$ is higher than the pressure $p_N(x)$ for the case of a lubricant with Newtonian rheology and for higher $\mu_p,\ \lambda$, and α_0 pressure $p(x)$ is higher.

For non-Newtonian fluids with polymeric additives described by the Giesekus model it makes sense also to consider some anisotropic fluid properties such as the first stress invariant which being scaled the same way as stresses (see (2.1)) in the dimensionless form is

$$N_1 = \tau_{xx} - \tau_{zz}. \tag{3.2}$$

Obviously, the value of $N_1(x, z)$ can be easily calculated using (2.22)-(2.24). A typical

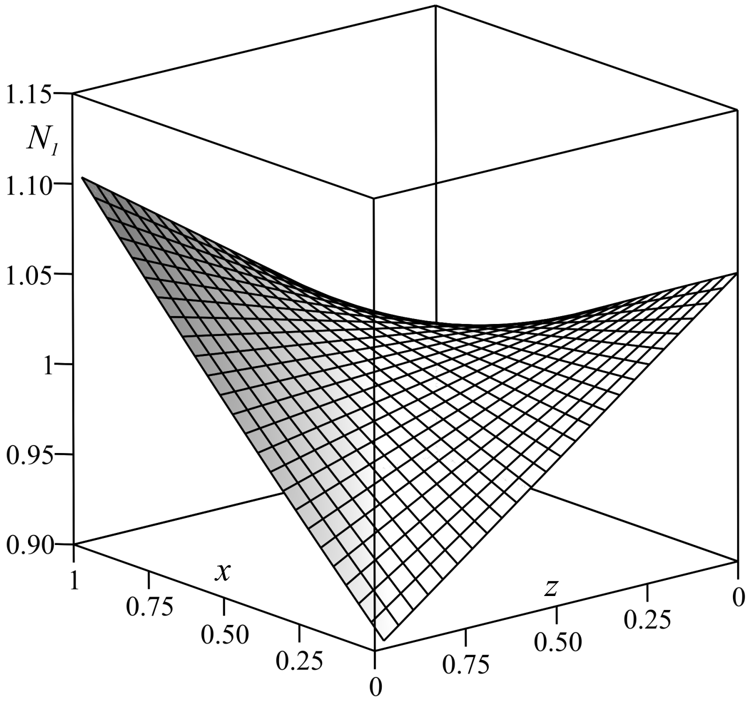

Fig. 3: A typical distribution of stress $N_1(x,z)$ obtained for
$\epsilon = 0.05,\ Re_0 = 5,\ \mu_p = 0.25,\ \lambda = 1,\ \alpha_0 = 2$, and $m_0 = -0.5$.

graph of $N_1(x,z)$ is presented in Fig. 4. The value of $N_1(x,z)$ is positive for all x and z values which means that the carrying load of a bearing lubricated by a fluid with the Giesekus rheology is higher than that for the same bearing lubricated by a Newtonian fluid.

In addition to that one can easily calculate the energy loss in the lubricated contact and the friction force created by the lubricant flow by using the expressions for the stress component $\tau_{zx}(x,z)$ and and the lubricant velocity distribution $u(x,z)$ obtained above.

4 Closure

A new relatively simple asymptotic modeling of the behavior of the Giesekus lubrication parameters in the line contact of a rigid thrust bearing was developed. The rheology equations of the lubricant were simplified using the scale analysis assuming that the size of the lubricant layer across it is much smaller than along it. The solutions of the simplified rheology equations were obtained in the form of power series in small parameter ϵ. That allowed for derivation of the Reynolds equations of the zero, first, and second orders and the application of a regular perturbation method which simplified the problem. The lubrication problems based on the Reynolds equations of the zero, first, and second orders were solved analytically. All hydrodynamic parameters of the contact such as pressure, shear stress, coefficient of friction, energy loss etc. were determined analytically.

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Information about authors:

Ilya I. Kudish - ASME Fellow, ILRIMA Consulting LLC, Millersburg, MI 49759, USA, email - ilyakudish@gmail.com,

Sergei S. Volkov - Laboratory of Functionally Graded and Composite Materials, Research and Education Center "Materials Don State Technical University, 1 Gagrina sq., 344001, Rostov-on-Don,

Andrey S. Vasiliev - Laboratory of Functionally Graded and Composite Materials, Research and Education Center "Materials Don State Technical University, 1 Gagrina sq., 344001, Rostov-on-Don, Russia.

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