ՀԱՅԱՍՏԱՆԻԳԻՏՈՒԹՅՈՒՆՆԵՐԻԱԶԳԱՅԻՆԱԿԱԴԵՄԻԱՅԻՏԵՂԵԿԱԳԻՐ
ИЗВЕСТИЯ НАЦИОНАЛЬНОЙ АКАДЕМИИ НАУК АРМЕНИИ

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# ON THE DERIVATION OF A STRING EQUATION

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**Keywords:** string equation, asymptotic methods, mathematical rigour.

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**К вопросу о выводе уравнения струны.**

**Ключевые слова:** уравнение струны, асимптотические методы, математическая строгость.

В работе пересматривается вывод классического волнового уравнения, описывающего поперечные колебания упругой струны. Предлагаемый подход базируется на математически корректном применении второго закона Ньютона в случае малого участка струны. Помимо этого, приводится асимптотическое решение плоской задачи теории упругости для предварительно деформированной полосы. Показано, что обсуждаемое одномерное волновое уравнение соответствует главному длинноволновому низкочастотному приближению двумерного решения. В то же время, в следующем приближении уравнение движения струны уже не является гиперболическим, вследствие появления дисперсионного члена с четвертой производной.

**Կապլունով Յու. Դ, Պրիկազչիկով Դ.Ա.**

**Լարի հավասարման արտածման մասին**

**Հիմնաբառեր՝** լարի հավասարում, ասիմպտոտիկ մեթոդ, մաթեմատիկական խստություն:

Աշխատանքում վերանայվում է լարի լայնական տատանման կլասիկ հավասարումների արտածումը: Առաջարկվող մոտեցումը հիմնվում է լարի փոքր տեղամասում Նյուտոնի երկրորդ օրենքի մաթեմատիկորեն խիստ կիրառման վրա: Բացի այդ բերվում է նախօրոք դեֆորմացված շերտի առաձգականության տեսության հարթ խնդրի ասիմպտոտիկ լուծումը: Ցույց է տրված, որ քննարկվող միաչափ ալիքային հավասարումը համապատասխանում է երկչափ լուծման ցածր հաճախության մոտարկմանը: Միևնույն ժամանակ հաջորդ մոտարկման լարի շարժման հավասարումը արդեն հիպերբոլիկ չէ, չորրորդ ածանցյալով դիսպերսիոն անդամի առաջացման պատճառով:

The traditional derivation of the wave equation for an elastic string is revised. The focus is on a rigorous implementation and subsequent analysis of the Second Newton's Law adapted for a small string element. Asymptotic treatment of the plane strain problem for a pre-stressed elastic strip shows that the 1D classical wave equation corresponds to the leading order long-wave low-frequency approximation. At the same time, the next order approximation is not given by a hyperbolic equation supporting a dispersive transverse motion.

**Introduction.** An elastic string is seemingly the most popular example in the textbooks on PDEs and mathematical physics, see e.g. [1,2,3,4], illustrating the derivation of the canonical hyperbolic wave equation. At the same time, even the best mathematicians, e.g. see the correspondence between A.D. Myshkis and O.A. Oleynik [5], are not quite comfortable with string analysis. Apparently, the point is that the underlying physical framework, including the assumption on a prescribed uniform tension, with its orientation varying in time and space, as well as peculiarities of the implementation of anintegral form of the Second Newton's Law, needs to be fully appreciated. It is also worth noting that the famous introductory texts on linear elasticity, see e.g. [6-9] usually do not consider a string, which is governed by a more elaborated linearized theory for pre-stressed elastic solids. On other hand, more specialized applied books on elastic waveguides, e.g. [10,11] oftenlack a mathematical rigour when dealing with a string.

Another fundamental issue is that a string as a physical object has a small but finite thickness, similarly to thin elastic rods, plates and shells. For the latter, the equations of motion are always established by the reduction of the original 3D equations of motion to lower dimensional models, e.g. see [12,13]. For a string, such reduction was developed for plane-strain deformation [14] and later extended to a membrane in [15]. The cited papers start from linearized equations for pre-stressed incompressible elastic solids [16].

Below we start from the “exact” formulation (within 1D context) of the equation of motion for a small string element. All the steps of the limiting process, including the evaluation of the curvilinear integral associated with the inertial term using the mean value theorem, are addressed in detail. In this case, the linearization leading to the sought for wave equation is performed at the very last stage.

In addition, we presenta brief revisit of the string problem, along with the lines of the consideration in [14]. It is demonstrated that the classical wave equation in case of a string is just the leading order long-wave low-frequency approximation of the associated plane-strain problem. A dispersive term with fourth-order derivative, arising at next order, enables smoothening the discontinuity at the characteristics of the leading order hyperbolic operator.

**1. One-dimensional derivation.** First, consider a traditional 1D model in the variables $x$ and $t$, assuming that the tension $T$ is uniform, and is always oriented along the tangent to the string profile given by the function $u = u(x, t)$. Another problem parameter is mass density per unit length $\rho$. Consider a small but finite string element of length $\Delta x$, see Fig. 1, where $\theta(x, t)$ is the angle formed by the tangent with the horizontal axis $x$. The element is assumed to be stretched by the tension $T$ at its ends.

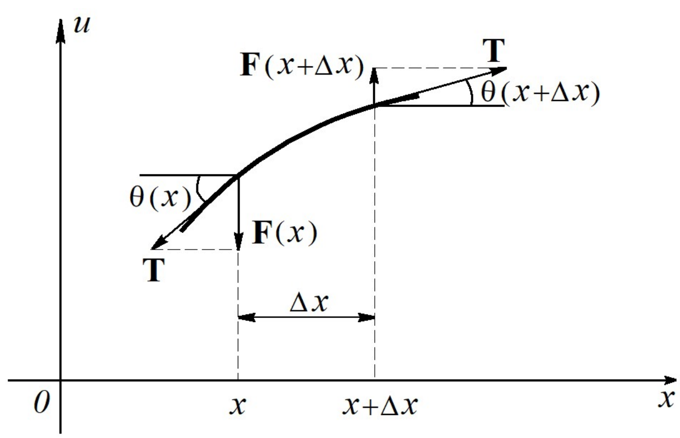


Fig. 1. String element under tension.

According to the Second Newton's Law, the equation of transverse motion of the chosen string element is given by

$$\Delta F = \rho \int_C \frac{\partial^2 u}{\partial t^2} ds, \tag{1}$$

where $C$ is the curve in $(x, u)$ coordinate frame defined by $x \leq \xi \leq x + \Delta x$, with

$$\Delta F = F(x + \Delta x, t) - F(x, t) = T\{sin(\theta(x + \Delta x, t)) - sin(\theta(x, t))\}. \tag{2}$$

Here we emphasize that the tension $T$ is the only force arising in the string. At the same time, its vertical projection varies in space and time, supporting transverse wave propagation.

The inertial term in (1) is expressed through a curvilinear integral of the first kind over the line segment of the string profile, hence we have from (1) and (2)

$$T\{sin(\theta(x + \Delta x, t)) - sin(\theta(x, t))\} = \rho \int_x^{x+\Delta x} \frac{\partial^2 u(\xi, t)}{\partial t^2} \sqrt{1 + \left(\frac{\partial u(\xi, t)}{\partial \xi}\right)^2} d\xi. \tag{3}$$

The formulation (3) is exact within the current 1D setup. To the best of authors' knowledge, this equation specifying the Second Newton's Law for a string does not usually appear in standard textbooks [1,2,11].

Let us simplify this equation for small $\Delta x$, assuming for the sake of definiteness that the function $u = u(x, t)$ is twice differentiable in $x$ and $t$. First, we have in the left hand side of (3)

$$\Delta F = 2Tsin\left(\frac{\theta(x + \Delta x, t) - \theta(x, t)}{2}\right) cos\left(\frac{\theta(x + \Delta x, t) + \theta(x, t)}{2}\right)$$

$$\approx 2Tsin\left(\frac{1}{2}\frac{\partial \theta(x, t)}{\partial x}\Delta x\right) cos(\theta(x, t)) \approx T\Delta x \frac{\partial \theta(x, t)}{\partial x} cos(\theta(x, t)). \tag{4}$$

Here we neglected the quantities of order $O((\Delta x)^2)$, since the function $\theta(x, t)$ is differentiable with respect to $x$.

Now recall that

$$\theta = tan^{-1}\left(\frac{\partial u}{\partial x}\right), \tag{5}$$

according to the definition of the tangent. Then, we have in the right hand side of (4)

$$\frac{\partial \theta}{\partial x} = \frac{\partial^2 u}{\partial x^2}\left(1 + \left(\frac{\partial u}{\partial x}\right)^2\right)^{-1}, \quad \text{and} \quad cos\theta = \left(1 + \left(\frac{\partial u}{\partial x}\right)^2\right)^{-1/2}. \tag{6}$$

As a result,

$$\Delta F \approx \frac{\partial^2 u}{\partial x^2}\left(1 + \left(\frac{\partial u}{\partial x}\right)^2\right)^{-\frac{3}{2}} \Delta x \tag{7}$$

Next, we have from the right hand side of (3), using the mean value theorem

$$\int_{x}^{x+\Delta x} \frac{\partial^2 u(\xi,t)}{\partial t^2} \sqrt{1+\left(\frac{\partial u(\xi,t)}{\partial \xi}\right)^2}\, d\xi = \left[\frac{\partial^2 u(x',t)}{\partial t^2}\left(1+\left(\frac{\partial u(x',t)}{\partial x'}\right)^2\right)^{\frac{1}{2}}\right]\Delta x \approx$$

$$\approx \left[\frac{\partial^2 u(x,t)}{\partial t^2}\left(1+\left(\frac{\partial u(x,t)}{\partial x}\right)^2\right)^{\frac{1}{2}}\right]\Delta x, \tag{8}$$

where $x' = x + \eta\Delta x,\ 0 < \eta < 1$. On substituting (7) and (8) into (3), we obtain a nonlinear equation given by

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \tag{9}$$

where $c^2 = T/\rho$ is the conventional squared speed within the string.

Finally, assuming the displacement gradient is small, i.e.$|\partial u/\partial x| \ll 1$, we obtain the classical linear equation of string motion

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}. \tag{10}$$

**2. Asymptotic derivation in two-dimensional case.** We start from plane-strain equations for a pre-stressed, incompressible elastic strip $-\infty < x < \infty$, $-h \le y \le h$, see Fig. 2,

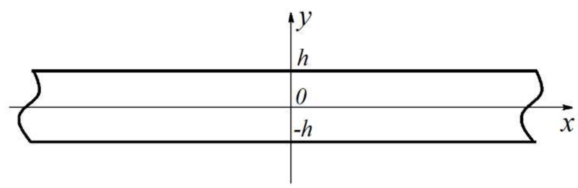


Fig. 2. Schematic of an elastic strip.

written in a symbolic form as

$$L[\boldsymbol{u}] = \rho\frac{\partial^2 \boldsymbol{u}}{\partial t^2}, \tag{11}$$

where $L$ is a 2x2 second order matrix linear differential operator and $\boldsymbol{u} = (u_1, u_2)$ is the displacement vector, satisfying the linearized measure of the incompressibility condition $\text{div}\boldsymbol{u} = 0$. Here and below see [14] for more specific details. The pre-stress is assumed to be in the form of horizontal tension $T$, uniform across the strip thickness. It should be noted that the parameters $T$ and $\rho$ have different dimensions than their counterparts in Section 1, which however does not affect the observations in the paper.

The homogeneous boundary conditions at traction-free faces$y = \pm h$ are given by

$$l_i[\boldsymbol{u}] = 0, \quad i = 1,2, \tag{12}$$

where $l_i$ are appropriate first-order differential operators.

Consider long-wave low-frequency motion of the strip, for which

$$x = \lambda\xi_1,\ \ y = h\xi_2,\ \ t = \frac{\lambda\tau}{c}, \tag{13}$$

where $\lambda-$ typical wave length, $c = \sqrt{T/\rho}$ . Here the ratio $\eta = h/\lambda \ll 1$ is assumed to be small. Next, the displacements are expanded into asymptotic series in terms of this parameter as

$$u_1 = \eta \sum_{i=0}^{\infty} u_{1i}\eta^{2i}, \qquad u_2 = \sum_{i=0}^{\infty} u_{2i}\eta^{2i}. \tag{14}$$

On substituting these series into the equations of motion and boundary conditions (11), (12), and the aforementioned incompressibility condition, expressed in the dimensionless variables$\xi_1, \xi_2$and $\tau$, we arrive at leading order to the wave equation (10), for $u = u_{20}$. Thus, 1D approach exposed in the previous section appears to be asymptotically justified.

At next order, we have for $u = u_{20} + \eta^2 u_{21}$ a refined equation given by

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} + \frac{h^2}{c^2}\left(\frac{1}{3} - \frac{\delta}{T}\right)\frac{\partial^4 u}{\partial x^2 \partial t^2} = 0, \tag{15}$$

in which the coefficient at senior derivative cannot be expressed only through the basic problem parameters $T$ and $\rho$, but also involves a more specific characteristic of pre-stress, denoted for brevity by $\delta$. The presence of such fourth-order dispersive term results in smoothening of the discontinuities predicted by the classical string equation (10). In this case, depending on the sign of the coefficient, the associated wave front can be either receding ($T > 3\delta$) or advancing ($T < 3\delta$), e.g. see Fig. 3 a) and b), respectively.

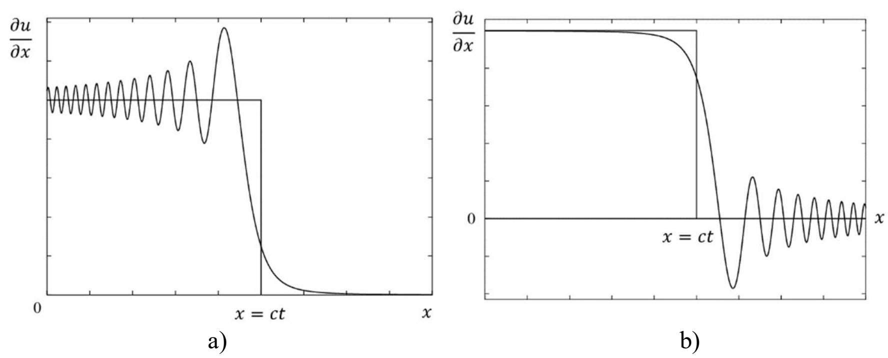


Fig. 3. Dispersive behaviour near the wave front $x = ct$: a) receding; b) advancing.

**3. Concluding remarks.** The 1D derivation presented in Section 1 starts with a rigorous formulation of the Second Newton's Law within adapted physical assumptions. Each step of the limiting process, as the size of the chosen string element tends to zero, is addressed in detail. Nevertheless, the initially nonlinear problem for a string does not seem to be the best example for illustrating the derivation of the wave equation. In particular,a simpler problem for the longitudinal waves in an elastic rod looks more preferable. Indeed, the latter does not involve relatively elaborated geometrical consideration, along with a nontrivial hypothesis regarding uniform tension tangent to the string profile, but operates with a lucid Hooke's Law instead.

The 2D analysis in Section 2 demonstrates that the fundamental wave equation (10) is not exact, but corresponds to a specific leading order long-wave approximation for transverse low-frequency motion for a thin pre-stressed elastic strip. In this case, the refined equation (15) involves a term with a fourth-order derivative resulting in dispersive wave propagation.

REFERENCES


1. Tikhonov A.N., Samarskii A.A. Equations of mathematical physics.// Courier Corporation. – 2013.
2. Vladimirov V.S. Equations of mathematical physics //Moscow, Nauka. – 1976.
3. Sveshnikov A.G., Bogolyubov, A.N., Kravtsov V.V. Lectures on mathematical physics // Moscow, Moscow State University. – 2004.
4. Petrovsky I.G. Lectures on partial differential equations. – Courier Corporation, 2012.
5. Myshkis A.D. Soviet Mathematicians: My Memories //Editorial LKI, Moscow. – 2007.
6. Nowacki W. Theory of elasticity //M.: Mir. – 1975.
7. Love A.E.H. A treatise on the mathematical theory of elasticity. – Cambridge university press, 2013.
8. Amenzade Y.A. Theory of elasticity // M.: Mir. – 1979.
9. Filonenko-Borodich M.M. Elasticity theory //Gostekhteorizdat, Moscow. – 1959.
10. Graff K.F. Wave motion in elastic solids. – Courier Corporation, 2012.
11. Coulson C.A. Waves-A mathematical account of the common types of wave motion //University of Oxford, England, Oliver and Boyd Interscience Publishers Inc., Edinburgh, Scotland. – 1955.
12. Gol'denveizer A.L. Theory of elastic shells // Elsevier – 2014.
13. Ambartsumyan S.A. General Theory of Anisotropic Shells// M.:Nauka. – 1974.
14. Kaplunov J.D., Nolde E.V., Rogerson G.A. A low-frequency model for dynamic motion in pre-stressed incompressible elastic structures //Proceedings of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences. – 2000. – T. 456. – №. 2003. – C. 2589-2610.
15. Pichugin A.V., Rogerson G.A. An asymptotic membrane-like theory for long-wave motion in a pre-stressed elastic plate //Proc. of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences.–2002. – T. 458. – №. 2022. – C. 1447-1468.
16. Ogden R.W. Nonlinear elastic deformations //Dover, Mineola. – 1984.



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