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ИЗВЕСТИЯ НАЦИОНАЛЬНОЙ АКАДЕМИИ НАУК АРМЕНИИ

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Devoted to the memory of professorMelsBelubekyan

LOCALISED WAVES IN ELASTIC THIN-WALLED STRUCTURES

K.Ghazaryan, P. Marzocca

Key Words: Plate, shell, bending edge wave, localisation, membrane theory of shell

Նվիրվում է պրոֆեսորՄելսԲելուբեկյանիհիշատակին

Կ. Ղազարյան, Պ. Մարզոկա
Տեղայնացվածալիքներըառաձգականբարակապատկառուցվածքներում

Հիմնաբառեր. Սալ, թաղանթ, ծռման եզրային ալիք, տեղայնացում, թաղանթների մեմբրանային տեսություն

Տվյալ հոդվածում հեղինակները ներկայացնում են բարակապատ կառուցվածքներում տեղայնացված ծռման ալիքներին առնչվող իրենց աշխատանքների ընդհանուր ամփոփումը: Հեղինակները ներկայացնում են նաև երկու նոր խնդիրների դրվածքն ու անալիտիկ արդյունքները՝ պայմանավորված ներդրակներով ամրանավորված համասեռ անվերջ սալում ծռման եզրային ալիքի տարածումով: Ամրանավորված ներդրակները մոդելավորված են որպես առաձգական հեծան, կամ առաձգական սալ:

Посвящается памяти профессора Мелса Белубекяна

Казарян К, Марзокка П.
Локализованные волны в упругих тонкостенных конструкциях

Ключевые слова: Пластинка, оболочка, изгибная краевая волна, локализация, мембранная теория оболочек

В данной статье авторы представляют обзор своих исследований, касающихся локализованных изгибных волн в тонкостенных конструкциях. Авторы также представляют постановку и аналитические результаты для двух новых задач, связанных с распространением изгибной краевой волны в однородной бесконечной пластине, армированной включениями, моделируемые как упругая балка или упругая пластина.

In this paper the authors provide a review of our investigations pertaining to localized bending waves in thin-walled structures. The authors also present the statements and analytical results fortwo new problems related to bending edge wave propagation in homogeneous infinite plate reinforced by inclusions, modelled as elastic beam or elastic plate.

1. Localized bending waves are perturbations concentrated in the vicinity of the free edge of thin plates and shells and decaying within a short distance from the edge. These bending localized waves are also called “edge waves” or “edge resonance waves”. Based on the Kirchhoff theory of isotropic elastic thin plates, the existence of a bending wave localised near the free edge of a semi-infinite medium was first demonstrated by Konenkov in [1].

The first edge waves' results published in English were documented in [2,3], where was rediscovered the same phenomenon concurrently and independently, without being aware of Konenkov's contribution.

From a mathematical point of view, the edge wave resonance eigenvalue problem is similar to the eigenvalue problem for the local stability of plate [4] which was firstly reported in [5].

The problem of bending waves localized near the free edge of a transversely isotropic plate is investigated in [6] using the Ambartsumian's higher-order plate theory which takes account of the transverse shears generated by flexural deformation. Unlike the first order Reissner–Mindlin theory, which also takes account of transverse shears, Ambartsumian's theory does not demand that plane normal cross-sections remain plane during bending. Within this analysis the existence of localized bending waves in transversely isotropic plates is established, and solutions of the dispersion equation obtained for different values of the elastic parameters. The analysis of frequencies of localized bending waves shows that for thick plates the effect of anisotropy can be considerable. For the case of vibrations of a narrow plate, from the long wave approximation a new beam vibration equation of the Timoshenko type is obtained for a transversally isotropic plate.

Within the framework of the Ambartsumian's higher-order plate theory, in [7] the existence and propagation problems of electro-elastic bending waves localized at the free edge of a 6mm hexagonal symmetry piezoelectric plate was established. The condition for existence of a localized bending wave is obtained, and the dispersion equation solved with respect to a dimensionless frequency. It is shown that the piezoelectric effect can increase the attenuation coefficient for a localized wave by up to 70% compared with that for a purely elastic plate, thus significantly decreasing the depth of penetration. The problem is also solved within the classical Kirchhoff theory. A comparison of results is carried out between two theories.

The study of planar and bending magnetoelastic vibrations of a perfectly conductive flat plate immersed in a uniform external magnetic field is presented in [8]. The Kirchhoff's plate theory and the model of a perfect conductive medium are used. The conditions for the existence of localized bending vibrations in the vicinity of the free edge of the plate are established. It is shown that the localized vibrations can be detected and can be eliminated by means of an applied magnetic field.

The problems of localized bending waves for elastic isotropic and orthotropic cantilever plates with a rib reinforcement were studied in [9,10]. Herein the effect of inertial and elastic contributions due to the rib have been separately analysed. These investigations revealed that the presence of a reinforcement rib can suppress localized bending waves.

In the framework of the membrane theory of cylindrical shells [11,12], the localised vibration near free edges of finite and semi-infinite cylindrical shell is considered. The derived dispersions equations lead to the localised membrane vibration conditions which are analysed and the appropriate recommendations are offered.

Localized magnetoelastic bending vibration of an electroconductive elastic plate [8]

The study of bending magnetoelastic vibrations of a perfectly conductive flat plate immersed in a uniform external magnetic field is presented. Kirchhoff's plate theory and the model of a perfect conductive medium are used. For this system it can be shown that localized bending vibrations exist in the vicinity of the plate free edge and can be detected and eliminated by means of changing the intensity magnitude of the magnetic field.

An elastic electroconductive plate is considered and is immersed in an external longitudinal magnetic field parallel to the (x, y) plane, (Fig. 1). $\mathbf{H_0} = H_{01}\mathbf{i_x} + H_{02}\mathbf{i_y}$ of constant intensity $\left(H_{01} = const\right.$, $\left.H_{02} = const\right)$.

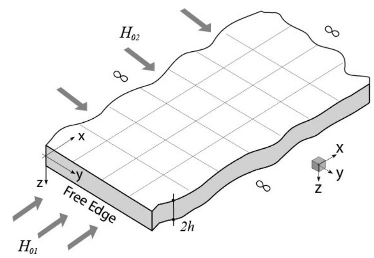

Fig.1 Model of a plate immersed in a magnetic field in x and y directions.

In framework of the Kirchhoff's theory and the model of a perfect conducting medium the plate vibration equation can be defined as

$$D\Delta^2 w - \frac{h}{2\pi}\left(H_{01}^2\frac{\partial^2 w}{\partial x^2} + H_{02}^2\frac{\partial^2 w}{\partial y^2} + 2H_{01}H_{02}\frac{\partial^2 w}{\partial x \partial y}\right) + 2\rho h\frac{\partial^2 w}{\partial t^2} = 0 \tag{1}$$

where w is the plate normal deflection.

Next the localized bending vibrations of a semi-infinite plate occupying the domain $0 \le x < \infty$, $-\infty < y < \infty$, $-h \le z \le h$ is considered. It is assumed that the localized waves are propagating along the y axis.

The associated boundary conditions at the edge $x = 0$ are:

$$\left(\frac{\partial^2 w}{\partial x^2} + \upsilon\frac{\partial^2 w}{\partial y^2}\right) = 0; \quad D\frac{\partial}{\partial x}\left[\frac{\partial^2 w}{\partial x^2} + (2-\upsilon)\frac{\partial^2 w}{\partial y^2}\right] - \frac{hH_{01}^2}{2\pi}\frac{\partial w}{\partial x} = 0 \tag{2}$$

At $x \to \infty$ the vibration damps out, implying that $\lim\limits_{x\to\infty} w = 0$.

The two special cases of a magnetic field are considered:

Magnetic field perpendicular to wave propagation

In this case $H_{01} \ne 0, H_{02} = 0$. The solution of Eq. (1) can be written as:

$$w(x, y) = w_0\left(C_1 e^{-kpx} + C_2 e^{-kqx}\right)e^{i(\omega t - ky)} \tag{3}$$

where k is the wave number and ω is the frequency of vibration,

$$p = \sqrt{\left(1 + \chi + \sqrt{\eta^2 + 2\chi + \chi^2}\right)}, \quad q = \sqrt{\left(1 + \chi - \sqrt{\eta^2 + 2\chi + \chi^2}\right)},$$

$$\eta^2 = \frac{2\rho h\omega^2}{Dk^4}, \chi = \frac{hH_0^2}{4\pi Dk^2}$$

The dimensionless parameter η is related to the frequency of localized vibration, and according to the condition of damping it should satisfy the inequality $0 < \eta^2 < 1$. The roots of the dispersion equation can be cast in the following form:

$$\eta^2 = 1 + 2(1 - v - \chi)\sqrt{(1 - v - \chi)^2 + v^2} - 2(1 - v - \chi)^2 - v^2 \tag{4}$$

Sufficient and necessary conditions for the existence of localized bending waves have the form

$\chi < (3+v)(1-v)/2, \quad v \neq 0$

In particular, for a steel plate $(E = 110\,\mathrm{GPa},\ \upsilon = 0.3)$, with relative wave-length of $kh = 0.02$ and $kh = 0.01$, a value of $\chi_0 = 1.15$ is needed to eliminate localized vibrations, resulting in an intensity of the magnetic field on the order of $H_{01} \sim 1.5T$ and $H_{01} \sim 0.37T$, respectively.

Magnetic field parallel to wave propagation

In this case $H_{01} = 0, H_{02} \neq 0$ and application of H_{02} leads to an increase in the localized vibration frequencies with increasing magnetic field intensity, implying that the localized vibrations exist, regardless of the magnitude of intensity of the magnetic field.

Orthotropic plate reinforced by a rigid rib [9]

A study of the localized bending wave in a thin elastic orthotropic cantilever plate reinforced by a rigid rib is presented. A general solution is given and the particular case of an isotropic reinforced plate is analyzed. The bending vibration equation issolved in conjunction with appropriate boundary conditions and an avenue to identify the rib elastic proprieties through an inverse approach is described.

A rectangular elastic plate in a Cartesian reference system (x, y, z) is considered such that the plane (xOy) coincides with the plate middle surface, with z as the coordinate along thickness of a plate, such as , $y \in [0, b]$, $z \in [-h, h]$ (Fig.2)

Based on the Kirhhoff's hypothesis, a plate bending vibration equation can be written as

$$D_{11}\frac{\partial^4 w}{\partial x^4} + 2(D_{11} + 2D_{66})\frac{\partial^4 w}{\partial x^2 \partial y^2} + D_{22}\frac{\partial^4 w}{\partial y^4} + 2\rho h\frac{\partial^2 w}{\partial t^2} = 0 \tag{4}$$

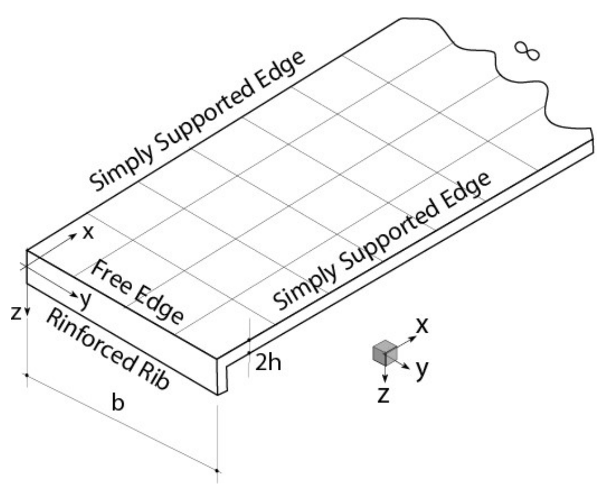

Figure 2: Model of a cantilever plate with one free and rib reinforced edge.

Here $w(x, y)$ is a plate middle surface normal displacement, $2h$ is the plate thickness, ρ is a density of a plate material, $D_{11}, D_{12}, D_{22}, D_{66}$ are physical constants characterizing plate rigidness.

On plate edges, e.g. $y = 0, b$, simply supported boundary condition has been assumed.

The edge $x = 0$ is supposed to be free from mechanical stresses and reinforced with rib, which is modeled as an elastic beam. On this edge, the following boundary conditions are applied

$$-D_{11}\frac{\partial^2 w}{\partial x^2} - D_{12}\frac{\partial^2 w}{\partial y^2} = A_0\frac{\partial}{\partial y}\left(\frac{\partial^2 w}{\partial x \partial y}\right),$$

$$-\frac{\partial}{\partial x}\left[D_{11}\frac{\partial^2 w}{\partial x^2} + \left(D_{12} + 2D_{66}\right)\frac{\partial^2 w}{\partial y^2}\right] = D_0\frac{\partial^4 w}{\partial y^4}$$

$$D_{11}\frac{\partial^2 w}{\partial x^2} + D_{12}\frac{\partial^2 w}{\partial y^2} + A_0\frac{\partial^3 w}{\partial x \partial y^2} = 0$$

$$\frac{\partial}{\partial x}\left[D_{11}\frac{\partial^2 w}{\partial x^2} + \left(D_{12} + 4D_{66}\right)\frac{\partial^2 w}{\partial y^2}\right] + D_0\frac{\partial^4 w}{\partial y^4} = 0$$

Herein, A_0 and D_0 are the twist and bending stiffness of the beam, respectively.

As a limiting case, if the plate is semi-infinite, the attenuation (localization) condition for the out of plane displacement at $x \to \infty$ is assumed as $\lim_{x\to\infty} w(x, y, t) = 0$

The necessary and sufficient condition of the existence of localized waves is found as

$$\gamma\beta\lambda_n^2 + \gamma\lambda_n\sqrt{2(\alpha_2 + \alpha_3)} - \alpha_1^2 < 0 \tag{5}$$

$$\alpha_1 = \frac{D_{22}}{D_{11}}, \quad \alpha_2 = \frac{D_{12}}{D_{11}}, \quad \alpha_3 = \frac{2D_{66}}{D_{11}}, \quad \beta = \frac{A_0}{D_{11}}, \quad \gamma = \frac{D_0}{D_{11}};$$

$$\lambda_n = \pi n/b, \qquad n = 1, 2, 3.....$$

Without a rib, i.e. $\gamma = \beta = 0$, the condition (5) always hold, while the presence of a rigid rib can eventually eliminate the localized wave if $\gamma\beta\lambda_n^2 + \sqrt{2}\gamma\lambda_n - \upsilon^2 < 0$.

On the other hand, for isotropic plate with rectangular square cross section rib the condition (5) can be written as

$$\frac{n^2\pi^2 a^8}{b^2h^6}\left(\frac{E_0}{E}\right)^2+\frac{\sqrt{2}\left(1-\upsilon^2\right)n\pi n a^4}{bh^3}\frac{E_0}{E}-\left(1-\upsilon^2\right)^2\upsilon^2<0$$

and the localized waves can be eliminated if

$$\frac{\pi a^4}{bh^3}\left(\frac{E_0}{E}\right)>\frac{\sqrt{2}}{2}\left(1-\upsilon^2\right)\left(\sqrt{1+2\upsilon^2}-1\right).$$

Effect of the stiffness and inertia of a rib reinforcement on localized bending waves in semi-infinite strips [10]

The problem of localized bending waves in an elastic semi-infinite plate with a rib reinforcement has been analyzed. The mathematical conditions for the existence of the waves have been derived from the equation of motion. In particular, the effects of inertial and elastic terms in the rib have been separately investigated, leading to an interesting duality. With such a configuration, the existence of localized bending waves for a massless rib reinforcement does not depend on the inertial properties of the strip. On the other hand, if only the inertia contributions of the rib are taken into account, the flexural stiffness of the strip plays no role. Results for several cross-sections and a typical aluminum alloy material have presented. Analyzing the changes in the dimensionless frequencies of the waves, it has been found that – for a circular and square cross sections – there exists a particular dimension for which the edge waves are equivalent to the edge waves occurring without any reinforcement. From the investigation of the regions of existence of the waves it appeared that the elastic contributions due to the rib are predominant compared to their inertial ones, at least for this problem. Furthermore, stiffer strips have been found to require smaller reinforcements to suppress their edge wave.

Localised vibrations of membrane cylindrical shell [11,12]

The problem of localized vibration has been studied in an elastic cylindrical shell using the equations of shell membrane theory. The shell has one edge which is traction free, while three different boundary conditions are considered on the other edge, namely, a clamped edge condition, and the Navier and anti-Navier boundary conditions. The corresponding dispersion equations have been obtained and analyzed to assess the existence of a localized vibration at the free-edges. For all boundary condition cases the qualitative behavior of dispersion curves is very similar for the selected values of the Poisson ratios. However, it is observed that the frequency decreases when the shell length increases, reaching asymptotic values more rapidly for higher wave numbers. It has also been shown that there are no qualitative differences between results of shells under Navier and anti-Navier edge conditions. For the case of a shell with a traction free edgeand clamped edge there is a minimum value of shell length/shell radius ratio where the localized vibration does not occur. In addition, for the shells with traction free edge/Navier and traction free edge/anti-Navier boundary conditions it is shown that localized vibrations occur for any shell length.

A cylindrical shell of length L , R radius of shell middle surface and thickness h (Fig.3) is considered next.

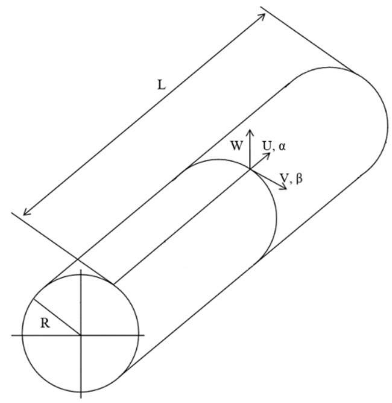

Fig.3 Cylindrical shell middle surface

The equations of motion and constitutive elastic law of cylindrical membrane shell are as follows

$$\Delta U+\theta_0\frac{\partial}{\partial\alpha}\left(\frac{\partial U}{\partial\alpha}+\frac{1}{R}\frac{\partial V}{\partial\beta}\right)+\frac{2v}{(1-v)R}\frac{\partial W}{\partial\alpha}=\frac{1}{c_t^2}\frac{\partial^2 U}{\partial t^2}$$

$$\Delta V+\frac{\theta_0}{R}\frac{\partial}{\partial\beta}\left(\frac{\partial U}{\partial\alpha}+\frac{1}{R}\frac{\partial V}{\partial\beta}\right)+\frac{2}{(1-v)R^2}\frac{\partial W}{\partial\beta}=\frac{1}{c_t^2}\frac{\partial^2 V}{\partial t^2} \quad (6)$$

$$\frac{1}{R}\frac{\partial V}{\partial\beta}+\frac{W}{R}+v\frac{\partial U}{\partial\alpha}=-\frac{\rho R h}{C}\frac{\partial^2 W}{\partial t^2}$$

Here $C=Eh\left(1-v^2\right)^{-1}$, ρ is the bulk density , E is Young's modulus , G is the shear modulus , v is the Poisson's ratio of the shell material, h is the shell thickness, U is the axial displacement along the generator , V is the circumferential displacement in the direction of the profile of the middle surface and W is the radial displacement normal to the middle surface, $\Delta\equiv\frac{\partial^2}{\partial\alpha^2}+\frac{1}{R^2}\frac{\partial^2}{\partial\beta^2}$, $c_t^2=\frac{G}{\rho}$, $\theta_0=\frac{1+v}{1-v}$,

When $\theta^{-1}n^{-2}<\eta^2<1$ the modes of all solutions of Eq. (6) have attenuated, non periodic, forms.

In the case of the boundary conditions of the clamped and traction free shell the particular case when the effect of bending deformation is negligible, namely when $\gamma=0$, the dispersion equation has the form

$$
\begin{aligned}
&4\left(2-\eta^2\right)^2\sqrt{s_1 s_2}+\left[4 s_1 s_2+\left(2-\eta^2\right)^2\right]\sinh\left(kp_1L\right)\sinh\left(kp_2L\right)- \\
&-\left[1+\left(2-\eta^2\right)^2\right]\sqrt{s_1 s_2}\cosh\left(kp_1L\right)\cosh\left(kp_2L\right)=0
\end{aligned}
\tag{7}
$$

where

$$
p_{1,2}=\frac{1}{\sqrt{2\left(1-v^2\gamma\right)}}\left[s_1+s_2-\gamma s_3\pm\sqrt{\left(s_1+s_2-\gamma s_3\right)^2-4\left(1-v^2\gamma\right)s_1\left(s_2-\gamma\right)}\right]^{1/2}
$$

$$
\theta=\frac{1-v}{2},\eta^2=\frac{\omega^2}{k^2c_t^2},\; k=\frac{n}{R},\; s_1=1-\eta^2,s_2=1-\theta\eta^2,
$$

$$
s_3=2-v^2\eta^2,\gamma=\left(1-\theta n^2\eta^2\right)^{-1}
$$

Herein ω is the circular frequency.
In the limit $\eta\to 1$ one can find

$$
4\sqrt{1-\theta}+kL\sinh\left(kL\sqrt{1-\theta}\right)-2\sqrt{1-\theta}\cosh\left(\sqrt{1-\theta}\,kL\right)=0
$$

which determines the critical length of shell $r_0=kL$, beyond which the localisation, edge resonance, occurs.

Taking into account that edge resonance take placeat $kL>>1$, the following condition for critical length is found

$$
kL>2\sqrt{1-\theta}\;\; or\;\; L/R>2\sqrt{1-\theta}\;/\,n
$$

In the general case when the effect of bending deformation is not negligible $\gamma\neq 0$, for sufficient long shell $r_0=kL$ the critical length of shell $r_0=kL$ is determined from the condition

$$
r_0=\frac{2p_0^2\left(1-v^2\gamma_0\right)+1+v-2v\gamma_0}{\left(1+v-2v\gamma_0\right)p_0}\left\{1-v^2\gamma_0+\frac{\left(1-\gamma_0\right)v}{1-\theta-\gamma_0}\left[1-\theta-\left(2-v^2\right)\gamma_0\right]\right\}
$$

where $p_0=\left[\dfrac{1-\theta-\gamma_0\left(2-v^2\right)}{2-\left(1-v^2\gamma_0\right)}\right]^{1/2}$, $\gamma_0=\dfrac{1}{1-\theta n^2}$

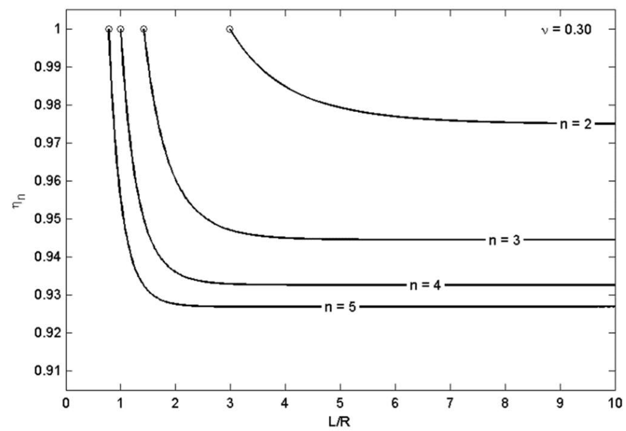

Fig.4 Dimensionless frequency η_n vs. L/R of the shell with traction free and clamped edges

From the dispersion curves, according to the wave number *n*, there is a minimum value of L/R under which localized waves do not take place. These values are reported in Table 1.

$(L/R)_{\min}$	$n=2$	$n=3$	$n=4$	
$\nu=0.20$	2.5970	1.3061	0.9246	0.7232
$\nu=0.25$	2.7848	1.3618	0.9602	0.7501
$\nu=0.30$	2.9872	1.4169	0.9947	0.7758
$\nu=0.35$	3.2083	1.4712	1.0279	0.8005

Table 1. Minimum values for L/R for the existence of localized waves corresponding to shell with traction free and clamped edges

The bending edge waves in an infinite elastic plate reinforced with rib

An infinite homogeneous elastic plate reinforced with inclusion, defined as an elastic rib, is considered next (Fig.5). The solutions for the edge wave attenuation due to the inclusion is to be found.

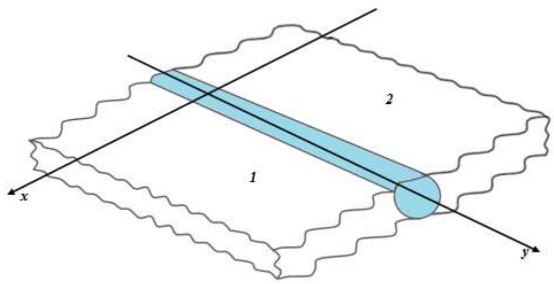

Fig.4 .Homogeneous elastic plate reinforced with an elastic rib .

The equations of plate vibration in $x > 0, s = 1$ and regions are

$$
\begin{aligned}
&D\left(\frac{\partial^4 w_s}{\partial x^4}+\frac{\partial^4 w_s}{\partial y^4}+2\frac{\partial^4 w_s}{\partial x^2 \partial y^2}\right)+2\rho h\frac{\partial^2 w_s}{\partial t^2} \qquad s=1,2\\
&\eta=\frac{\omega}{k^2}\sqrt{\frac{2\rho h}{D}};\ p=\sqrt{1-\eta};\ q=\sqrt{\eta+1};
\end{aligned} \tag{8}
$$

Solutions of Eq. (8) attenuated at $x \to \pm\infty$ can be written as

$$
\begin{aligned}
w_1 &= \left(C_1 \exp\left[-kpx\right]+C_2\exp\left[-kqx\right]\right)\exp\left[i\left(ky-\omega t\right)\right];\\
w_2 &= \left(C_3 \exp\left[kpx\right]+C_4\exp\left[kqx\right]\right)\exp\left[i\left(ky-\omega t\right)\right];
\end{aligned}
$$

Boundary conditions at $x = 0$ can be written as

$$
\begin{aligned}
&D_0\left(\left(\frac{\partial^2 w_1}{\partial x^2}-\frac{\partial^2 w_2}{\partial x^2}\right)+\upsilon\frac{\partial^2}{\partial y^2}\left(w_1-w_2\right)\right)+G_0 I_t\frac{\partial^3 w_2}{\partial x \partial y^2}-\rho_0 I_p\frac{\partial^3 w_2}{\partial x \partial t^2}=0\\
&D_0\left(\frac{\partial^3 w_2}{\partial x^3}\left(w_1-w_2\right)+\left(2-\upsilon\right)\frac{\partial^3}{\partial x \partial y^2}\left(w_1-w_2\right)\right)+E_0 J\frac{\partial^4 w_2}{\partial y^4}+\rho_0 S\frac{\partial^2 w_2}{\partial t^2}=0
\end{aligned} \tag{9}
$$

$$
\left(w_1-w_2\right)=0;\ \frac{\partial}{\partial x}\left(w_1-w_2\right)=0
$$

Here G_0, ρ_0, are shear modulus, bulk density of the rib material, I_t, I_p, J, S are cross-sectional torsional moment of inertia, cross-sectional polar moment of inertia, cross-sectional bending moment of inertia, cross-sectional area of the rib, correspondingly.

Substituting Eq. (10) into boundary conditions anhomogeneous set of algebraic equations can be found. These will have arbitrary constants, namely C_1, C_2, C_3, C_4. Equating the determinant of the set of equations to zero, the dispersion equations expressed in terms of the dimensionless frequency η can be found dispersion equations

$$\left(GI_t k^2 + 2Dk\left(p+q\right) - I_p \rho_0 \omega^2\right) = 0$$

$$\left(E_0 J k^4 + 2Dk^3 pq\left(p+q\right) - S\rho_0 \omega^2\right) = 0$$

The first equation can be rewritten in dimensionless notations as

$$\theta + \sqrt{1+\eta} + \sqrt{1-\eta} - \beta\eta^2 = 0;$$

$$\theta = \frac{G_0 I_t k}{2D}; \beta = \frac{I_p k^3 \rho_0}{4h\rho};$$

This equation has a solution $\eta < 1$ corresponding to the localized wave if $\beta > \sqrt{2} + \theta$
The second equation can be rewritten as

$$\alpha + \sqrt{1-\eta^2}\left(\sqrt{1-\eta} + \sqrt{1+\eta}\right) - \vartheta\eta^2 = 0$$

$$\alpha = \frac{E_0 J k}{2D}; \vartheta = \frac{kS\rho_0}{4h\rho}$$

For this equation the solution of the localized wave exists if $\vartheta > \alpha$.
From these localization conditions it follow that the rib inertia terms $S\rho_0$ and $\rho_0 I_p$ provide the existence of localized bending waves in plate.

The bending edge waves in a bi-material compound plates

Two semi-infinite plates of the same material (extend between $a \le y < \infty$ and $-\infty < y \le -a$) reinforced by elastically bonded finite plate of other material ($y < |a|$) are considered next. The finite plate is distinguished by an index (0) (Fig.5).

$$D_s\left(\frac{\partial^4 w_s}{\partial x^4} + \frac{\partial^4 w_s}{\partial y^4} + 2\frac{\partial^4 w_s}{\partial x^2 \partial y^2}\right) + 2\rho_s h \frac{\partial^2 w_s}{\partial t^2} = 0; s = 0,1,2 \qquad (10)$$

$$\rho_1 = \rho_2 = \rho; \quad D_1 = D_2 = D; \quad \nu_1 = \nu_2 = \nu$$

$$p = \sqrt{1-\eta}; \; q = \sqrt{1+\eta}; \; \eta = \frac{\omega}{k^2}\sqrt{\frac{2\rho_0 h}{D_0}};$$

Here w_s is the plate middle surface normal displacement, D_s are plate flexural rigidities, ρ_s are bulk densities of plate materials, $2h$ plate thickness.

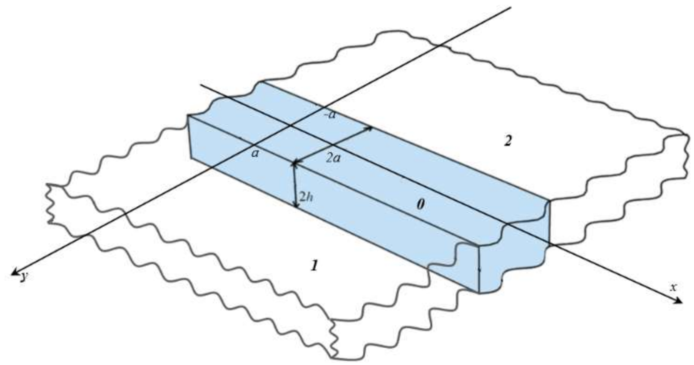

Fig.5 Semi-infinite plates of the same material (extend between $a \le y < \infty$ and $-\infty < y \le -a$) reinforced by elastically bonded finite plate

Solutions of (1) can be written as

$$w_0 = w(y)\exp\left[i(kx-\omega t)\right]; \quad y \in (-a,a)$$

$$w_1 = \left(B_1 \exp\left[-kp(y-a)\right] + B_2 \exp\left[-kq(y-a)\right]\right)\exp\left[i(kx-\omega t)\right]; y \in (a,\infty)$$

$$w_2 = \left(A_1 \exp\left[kp(y+a)\right] + A_2 \exp\left[kq(y+a)\right]\right)\exp\left[i(kx-\omega t)\right]; y \in (-a,-\infty)$$

The condition $\eta < 1$ condition prescribes wave attenuation at $y \to \pm\infty$.
The following continuity conditions apply

$$w_1 = w_0, \frac{\partial w_1}{\partial y} = \frac{\partial w_0}{\partial y} \qquad y = a; \qquad (11)$$
$$M_1 = M_0, N_1 = N_0;$$

$$w_2 = w_0; \frac{\partial w_2}{\partial y} = \frac{\partial w_0}{\partial y}; \qquad y = -a; \qquad (12)$$
$$M_2 = M_0 \;; N_2 = N_0$$

$$M_s = D_s\left(\frac{\partial^2 w_s}{\partial y^2} + \upsilon_s \frac{\partial^2 w_s}{\partial x^2}\right); N_s = D_s\left(\frac{\partial^3 w_s}{\partial y^3} + (2-\upsilon_s)\frac{\partial^3 w_s}{\partial y \partial x^2}\right)$$

Herein M_s, N_s are stress couples and generalized normal stress resultants, and υ_s are Poisson ratios of plate materials, respectively.

Substituting w_1 and w_2 into the first row of boundary conditions (11) and (12) one obtains

$$
\begin{aligned}
B_1 &= -\frac{kqw(a)+w'(a)}{k(p-q)}, \qquad B_2 = \frac{kpw(a)+w'(a)}{k(p-q)} \\
A_1 &= -\frac{kqw(-a)-w'(-a)}{k(p-q)}, \quad A_2 = \frac{kpw(-a)-w'(-a)}{k(p-q)};
\end{aligned}
\tag{13}
$$

Substituting w_1 and w_2 into second row of boundary conditions (11) and (12) and using (13) the boundary conditions at middle plate edges $y = \pm a$ canbe found:

$$k^2\left(pq\gamma+\gamma\upsilon-\upsilon_0\right)w(a)+k(p+q)\gamma w'(a)+w''(a)=0$$

$$k^3pq(p+q)\gamma w(a)+k^2\left(2-\upsilon+\gamma\left(-2+p^2+pq+q^2+\upsilon\right)\right)w'(a)-w'''(a)=0$$

$$k^2\left(pq\gamma+\gamma\upsilon-\upsilon_0\right)w(-a)-k(p+q)\gamma w'(-a)+w''(-a)=0,$$

$$k^3pq(p+q)\gamma w(-a)-k^2\left(2-\gamma\left(2-p^2-pq-q^2-\upsilon\right)-\upsilon_0\right)w'(-a)+w'''(-a)=0$$

$$\gamma = D/D_o; \qquad (\ )' = \frac{d}{dy}$$

The equation of middle plate and boundary conditions leads to the separate anti-symmetric or symmetric solutions:

Anti-symmetric solution

$$w(x) = c_1\sin\left(kp_0y\right)+c_2\sinh\left(kq_0y\right); \tag{14}$$

$$p_0=\sqrt{\eta_0-1}; \qquad q_0=\sqrt{\eta_0+1};\ \eta_0=\frac{\omega}{k^2}\sqrt{\frac{2\rho_0h}{D_0}};$$

Substituting Eq. (14) into boundary conditions an homogeneous set of algebraic equations, with respect to the arbitrary constants can be found. Equating the determinant of the set of equations to zero, the following dispersion equation determining frequency ω is also found:

Dispersion equation

$$
\begin{aligned}
&q_0\left(s+p_0^2f+q_0^2g-p_0^2q_0^2\right)\mathrm{tg}\left(akp_0\right)-p_0\left(s-p_0^2g-q_0^2f-p_0^2q_0^2\right)\mathrm{th}\left(akq_0\right)+ \\
&+\gamma(p+q)\left(p_0^2+q_0^2\right)\left(p_0q_0+pq\,\mathrm{tg}\left(akp_0\right)\mathrm{th}\left(akq_0\right)\right)=0
\end{aligned}
\tag{15}
$$

By the same way we can obtain the dispersion equation for symmetric solution

$$w(y) = c_3 \cos(kp_0 y) + c_4 \cosh(kq_0 y);$$

Dispersion equation

$$q_0\left(s + p_0{}^2 f + q_0{}^2 g - p_0{}^2 q_0{}^2\right)\mathrm{th}(akq_0) + p_0\left(s - p_0{}^2 g - q_0{}^2 f - p_0{}^2 q_0{}^2\right)\mathrm{tg}(akp_0) +$$

$$+ pq(p+q)\left(p_0{}^2 + q_0{}^2\right)\gamma - p_0 q_0 (p+q)\left(p_0{}^2 + q_0{}^2\right)\gamma\,\mathrm{tg}(akp_0)\,\mathrm{th}(akq_0) = 0 \quad (16)$$

In (15,16) the following notations are used

$$s = -(\gamma\upsilon - \upsilon_0)\left(2 + \gamma\left(-2 + q^2 + \upsilon\right) - \upsilon_0\right) +$$

$$+2pq\gamma(-1 + \gamma - \gamma\upsilon + \upsilon_0) + p^2\gamma\left(q^2\gamma - \gamma\upsilon + \upsilon_0\right);$$

$$f = \left(2 + \gamma\left(-2 + p^2 + pq + q^2 + \upsilon\right) - \upsilon_0\right);\, g = pq\gamma + \gamma\upsilon - \upsilon_0$$

The numerical analysis of dispersion equations (15,16), can provide practical recommendations. Two types of compound plate materials will be considered, namely $\rho_0/D_o \geq \rho/D$ and $\rho_0/D_o < \rho/D$.

Conclusions

This paper offers a detailed review of the authors' published investigations pertaining to localized magnetoelastic bending vibration of an electroconductive elastic plate, edge bending waves in orthotropic plate and isotropic strip reinforced by a rigid rib and localized vibration of cylindrical membrane shells. The authors also present new analytical results concerning bending wave localization at inclusion in infinite homogeneous plate modelled as beam or plate.

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Information about authors

Karen Ghazaryan ,Professor ,Principal Researcher, Institute of Mechanics National Academy of Sciences, Armenia,
Email : ghakrren@gmail.com

Pier Marzocca, Professor, Director, Sir Lawrence WackettDefence and Aerospace Centre,RMIT University, Australia,
Email:pier.marzocca@rmit.edu.au

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