ՀԱՅԱՍՏԱՆԻ ԳԻՏՈՒԹՅՈՒՆՆԵՐԻ ԱԶԳԱՅԻՆ ԱԿԱԴԵՄԻԱՅԻ ՏԵՂԵԿԱԳԻՐ
PROCEEDINGS OF THE NATIONAL ACADEMY OF SCIENCES OF ARMENIA

Մեխանիկա 75, № 1-2, 2022 Mechanics

UDC 539.3 Doi- 10.54503/0002-3051-2022.75.1-2-124

AVERAGED CONTROLLABILITY OF TRANSVERSELY ISOTROPIC AMBARTSUMYAN PLATE

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Keywords. particular theory of S. A. Ambartsumyan, refined theories of anisotropic plates, Green's function approach, infinite system

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Управляемость в среднем трансверсально изотропной пластинки Амбарцумяна

Ключевые слова: частная теория С. А. Амбарцумяна, уточненные теории анизотропных пластин, метод функции Грина, бесконечная система

В этой статье рассмотрена управляемость в среднем пластинки Амбарцумяна, изготовленной из трансверсально изотропного материала. Уравнения состояния основаны на гипотезах частной теории анизотропных пластин, разработанной С. А. Амбарцумяном для описания деформированного состояния анизотропных пластин, в каждой точке которой имеется плоскость изотропии, параллельная срединной плоскости пластинки. Применен метод функции Грина для явного представления нормального перемещения пластинки через материальные параметры пластинки (плотность, модули Юнга в обоих направлениях изотропии), являющиеся равномерно распределенными случайными величинами. В результате, условие управляемости в среднем сведено к бесконечной системе линейных уравнений относительно искомой функции управления. Построены три параметрических класса частных (эвристических) решений урезанного варианта бесконечной системы. Определение параметров управления сведено к решению задачи нелинейного программирования.

Ա. Ս. Ավետիսյան, Ս. Հ. Ջիլավյան, Աս. Ժ. Խուրշուդյան

Համբարձումյանի մասնակի ճշգրտված տեսությամբ նկարագրվող տրանսվերսալ իզոտրոպ սալի միջինացված ղեկավարելիությունը

Հիմնաբառեր՝ Ս. Ա. Համբարձումյանի մասնակի ճշգրտված տեսություն, անիզոտրոպ սալերի ճշգրտված տեսություններ, Գրինի ֆունկցիայի եղանակ, անվերջ համակարգ

Այս աշխատանքում հետազոտվում է Համբարձումյանի՝ տրանսվերսալ իզոտրոպ նյութից պատրաստ–ված սալի միջինացված ղեկավարելիությունը: Սալի ծռման հավասարումները ստացվում են անիզոտրոպ սալերի մասնակի ճշգրտված տեսության վարկածների հիման վրա, որը մշակել է Ս. Ա. Համբարձումյանը՝ սալի ամեն կետում միջին հարթությանը զուգահեռ իզոտրոպիայի հարթություն ունեցող անիզոտրոպ սալերի դեֆորմացված վիճակը նկարագրելու համար: Սալի նորմալ տեղափոխության բացահայտ կախվածությունը նյութի պարամետրերից (խտություն, իզոտրոպիայի երկու ուղղությամբ առաձգականության գործակիցներ) ստացվում է Գրինի ֆունկցիայի եղանակի կիրառմամբ: Ենթադրվում է, որ այդ պարամետրերը հավասարաչափ բաշխված պատահական մեծություններ են: Արդյունքում՝ միջինացված ղեկավարելիության պայմանից ստացվում է ղեկավարման որոնվող ֆունկցիայի նկատմամբ գծային հավասարումների անվերջ համակարգ: Կառուցվում է անվերջ համակարգի հատած մասի մասնավոր (էվրիստիկ) լուծումների երեք պարամետրական դաս, իսկ ղեկավարման պարամետրերի որոշումը հանգեցվում է ոչ գծային ծրագրավորման խնդրի լուծմանը:

In this paper we study the averaged controllability of Ambartsumyan plate made of a transversely isotropic material. Governing equations are based on assumptions of particular theory of anisotropic plates developed by S. A. Ambartsumyan specifically for describing the deformation of anisotropic plates made of material with isotropy plane which, at each point of the plate, is parallel to its middle plane. The Green's function approach is applied to express the plate normal displacement by means of material parameters (density, Young moduli in both directions of isotropy) which are assumed to be uniformly distributed random variables. Eventually, the averaged controllability condition is reduced to an infinite system of linear constraints with respect to the control action. Three types of particular (heuristic) solutions of the truncated version of the infinite system are discussed. Determination of control parameters is reduced to the solution of a problem of nonlinear programming.

1 Introduction

The concept of averaged controllability has been introduced by Enrique Zuazua in the recent paper [1] and has been further developed in [2–9] (also, refer to the list of references in [8]). Averaged controllability is an important criterion of controllability for systems or processes containing random parameters. In case of systems or processes described by partial differential equations, material characteristics may be (and most of the cases are) regarded as such parameters. The advantage of this notion is that controls providing the desired state exactly or approximately, do not depend on the random parameters. For instance, let a controlled state be described by $w\left(x,t,u;\alpha\right)$ where x and t are deterministic variables, u is the control and α is some random variable. The aim is to provide a desired state $w_T\left(x\right)$ at $t=T$. Then we will say that it is controllable in average if the controllability residue

$$\mathcal{R}_T\left(u\right)=||\bar{w}\left(x,T,u\right)-w_T\left(x\right)||$$

satisfies

$$\mathcal{R}_T\left(u\right)=0\quad\text{(exactly)}\quad\text{or}\quad\mathcal{R}_T\left(u\right)\leq\varepsilon\quad\text{(approximately)}$$

for some given ε and appropriate norm $||\cdot||$. Here, $\bar{w}$ is the averaged state given by

$$\bar{w}\left(\cdot,\cdot,\cdot\right)=\int_{\alpha_0}^{\alpha_1}w\left(\cdot,\cdot,\cdot;\alpha\right)p\left(\alpha\right)\mathrm{d}\alpha,$$

p is the PDF and α_0, α_1 are the extreme values of α.

Thence, substituting $\bar{w}$ into the definition of $\mathcal{R}_T\left(u\right)$, we will be able to derive a constrained on u that does not depend on α explicitly. It rather depends on the extreme values α_0 and α_1.

Generally, the analysis of exact averaged controllability is considerably simplified when the integral in the expression of $\bar{w}$ is explicitly evaluated. Nevertheless, in case when the integral can not be explicitly evaluated, approximate analytical expressions like trapezoidal or Simpson rule can work out with high accuracy. On the other hand, the analysis of approximate averaged controllability is often easier considering that the integral can be well estimated in terms of known integral inequalities.

In this paper, we consider a similar problem for a square plate simply supported at its four edges. The plate occupies the domain $\Omega = \{ (x, y, z) \in \mathbb{R}^3, (x, y) \in [0, l]^2, 2z \in [-h, h] \}$ and it is subject to a dynamical load $F(x, y, t) = u(t)\, v(x, y)$ with controllable u and given distribution function v. We will assume that at $t = 0$, the plate was resting in equilibrium. The plate is made of a transversely isotropic material. More specifically, at each point of the plate the isotropy plane is parallel to the middle plane of the plate.

2 Ambartsumyan hypotheses and governing equation

Denote by $\boldsymbol{w} = (w_x, w_y, w_z)$ the displacement field in the plate. Let us assume that the following hypotheses of Ambartsumyan particular theory are valid [10]:

1. w_z does not depend on z-coordinate, i.e., $w_z = w(x, y, t)$,
2. the normal stress σ_{zz} and the shear stresses τ_{xz}, τ_{yz} are determined according to classical anisotropic plate theory.
3. in the Hooke law, σ_{zz} is negligible with respect to the stresses σ_{xx}, σ_{yy} and τ_{xy}.

Note that in [11] it has been shown that Ambartsumyan theory of anisotropic plates is of fourth order similar to the linearized von Kármán equations. An optimal control problem for Ambartsumyan layer-plate has been considered in [12].

On the basis of these hypotheses, the displacement field will be expressed as follows:

$$w_x(x, y, z, t) = -z\frac{\partial w}{\partial x} + \frac{z}{2G'}\left(\frac{h^2}{4} - \frac{z^2}{3}\right)\varphi_0(x, y, t),$$

$$w_y(x, y, z, t) = -z\frac{\partial w}{\partial y} + \frac{z}{2G'}\left(\frac{h^2}{4} - \frac{z^2}{3}\right)\psi_0(x, y, t),$$

and the normal stress is determined from the third equation of motion of the classical theory subject to the boundary conditions

$$\sigma_{zz}\big|_{2z=h} = F(x, y, t), \quad \sigma_{zz}\big|_{2z=-h} = 0,$$

as follows:

$$\sigma_{zz} = \left(\frac{1}{2} + \frac{3}{2}\frac{z}{h} - 2\frac{z^3}{h^3}\right) F(x, y, t) + \rho \cdot \left(\frac{2z^2}{h^2} - \frac{1}{2}\right) z\frac{\partial^2 w_0}{\partial t^2}.$$

Here,

$$\varphi_0(x, y, t) = -\frac{E}{1-\nu^2}\frac{\partial \Delta w_0}{\partial x}, \quad \psi_0(x, y, t) = -\frac{E}{1-\nu^2}\frac{\partial \Delta w_0}{\partial y},$$

(ρ, E, ν, G') are the independent material parameters of the plate, Δ is the 2D Laplace operator, and w_0 is the normal displacement of the plate mid-plane calculated by the classical theory, i.e.,

$$D\Delta\Delta w_0 + \rho h \frac{\partial^2 w_0}{\partial t^2} = F(x, y, t) \quad \text{in} \quad \Omega, \tag{2.1}$$

where

$$D = \frac{Eh^3}{12(1-\nu^2)}$$

is the bending stiffness of the plate.

Then, the anisotropic normal displacement w of the plate mid-plane will satisfy the fourth order equation

$$D\Delta\Delta w + \rho h \frac{\partial^2 w}{\partial t^2} = F_1(x, y, t) \quad \text{in} \quad \Omega, \tag{2.2}$$

with

$$F_1(x, y, t) = F(x, y, t) - \frac{h^2}{10(1-\nu^2)} \frac{E}{G'} \left[\Delta F - \rho h \frac{\partial^2 \Delta w_0}{\partial t^2} \right],$$

subject to boundary conditions

$$\begin{cases} w = 0, \quad M_{xx} = 0 \quad \text{at} \quad x = 0, l \\ w = 0, \quad M_{yy} = 0 \quad \text{at} \quad y = 0, l. \end{cases} \tag{2.3}$$

Here, M_{xx} and M_{yy} are the in-plane bending moments given by

$$M_{xx}(x, y, t) = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right) + \frac{h^2}{10} \frac{D}{G'} \left(\frac{\partial \varphi_0}{\partial x} + \nu \frac{\partial \psi_0}{\partial y}\right),$$

$$M_{yy}(x, y, t) = -D\left(\nu \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) + \frac{h^2}{10} \frac{D}{G'} \left(\nu \frac{\partial \varphi_0}{\partial x} + \frac{\partial \psi_0}{\partial y}\right).$$

Note that (2.1) can be explicitly solved for various boundary conditions: refer to [13] for more details. Since in our case the plate is simply supported at its four edges, we will have [13]

$$w_0(x, y, t) = \frac{1}{\rho h} \int_0^t \int_0^l \int_0^l F(\xi, \eta, \tau)\, G(x, \xi, y, \eta, t-\tau)\, \mathrm{d}\xi \mathrm{d}\eta \mathrm{d}\tau, \tag{2.4}$$

$$G(x, \xi, y, \eta, t) = \frac{4}{l^2} \sqrt{\frac{\rho}{E}} \cdot \sum_{n,m=1}^{\infty} \frac{1}{\lambda_{nm}} \sin\left(\sqrt{\frac{E}{\rho}} \cdot \lambda_{nm} t\right) \varphi_n(x)\, \varphi_n(\xi)\, \varphi_m(y)\, \varphi_m(\eta),$$

$$\lambda_{nm} = \frac{h}{l^2} \frac{\pi^2 (n^2 + m^2)}{\sqrt{12(1-\nu^2)}}, \quad \varphi_n(x) = \sin\left(\pi n \frac{x}{l}\right).$$

Similarly, the general solution to (2.2) and (2.3) will be

$$
\begin{aligned}
w(x,y,t) = \frac{1}{\rho h}\int_0^t\int_0^l\int_0^l F_1(\xi,\eta,\tau)\,G(x,\xi,y,\eta,t-\tau)\,\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\tau = w_0(x,y,t) - \\
-\frac{h}{10(1-\nu^2)}\frac{E}{\rho G'}\int_0^t\int_0^l\int_0^l \Delta F\cdot G(x,\xi,y,\eta,t-\tau)\,\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\tau + \\
+\frac{h^2}{10(1-\nu^2)}\frac{E}{G'}\int_0^t\int_0^l\int_0^l \frac{\partial^2\Delta w_0}{\partial\tau^2}\cdot G(x,\xi,y,\eta,t-\tau)\,\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\tau.
\end{aligned}
$$

Denoting

$$
\frac{h}{10(1-\nu^2)}\frac{E}{\rho G'}\int_0^t\int_0^l\int_0^l \Delta F\cdot G(x,\xi,y,\eta,t-\tau)\,\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\tau = w_1(x,y,t;\boldsymbol{P}),
$$

$$
\frac{h^2}{10(1-\nu^2)}\frac{E}{G'}\int_0^t\int_0^l\int_0^l \frac{\partial^2\Delta w_0}{\partial\tau^2}\cdot G(x,\xi,y,\eta,t-\tau)\,\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\tau = w_2(x,y,t;\boldsymbol{P}),
$$

we will obtain

$$
w(x,y,t;\boldsymbol{P}) = w_0(x,y,t;\boldsymbol{P}) - w_1(x,y,t;\boldsymbol{P}) + w_2(x,y,t;\boldsymbol{P}).
$$

The indication of $\boldsymbol{P}$ in the argument of w and w_0, w_1, w_2 is merely to show their dependence on the material parameters.

2.1 Simplification of the displacement

Before proceeding with controllability problem formulation, let us first simplify w_0, w_1 and w_2 further. First, notice that

$$
w_0(x,y,t;\boldsymbol{P}) = \sum_{n,m=1}^{\infty} v_{nm}^0\varphi_n(x)\,\varphi_m(y)\cdot\int_0^t u(\tau)\,K_{nm}^0(t-\tau;E,\rho)\,\mathrm{d}\tau,
$$

$$
v_{nm}^0 = \frac{4}{l^2 h\lambda_{nm}}\int_0^l\int_0^l v(\xi,\eta)\,\varphi_n(\xi)\,\varphi_m(\eta)\,\mathrm{d}\xi\mathrm{d}\eta,
$$

$$
K_{nm}^0(t;E,\rho) = \frac{1}{\sqrt{\rho E}}\sin\left(\sqrt{\frac{E}{\rho}}\cdot\lambda_{nm}t\right),
$$

$$
w_1(x,y,t;\boldsymbol{P}) = \sum_{n,m=1}^{\infty} v_{nm}^1\varphi_n(x)\,\varphi_m(y)\cdot\int_0^t u(\tau)\,K_{nm}^1(t-\tau;\boldsymbol{P})\,\mathrm{d}\tau,
$$

$$
v_{nm}^1 = \frac{4h}{10l^2(1-\nu^2)\cdot\lambda_{nm}}\int_0^l\int_0^l \Delta v\cdot\varphi_n(\xi)\,\varphi_m(\eta)\,\mathrm{d}\xi\mathrm{d}\eta,
$$

$$
K_{nm}^1(t;\boldsymbol{P}) = \frac{1}{G'}\sqrt{\frac{E}{\rho}}\sin\left(\sqrt{\frac{E}{\rho}}\cdot\lambda_{nm}t\right).
$$

Further, notice that since $K_{nm}^{0}(0;\cdot,\cdot)\equiv 0$, we have

$$\frac{\partial^2}{\partial t^2}\int_0^t u(\tau)\,K_{nm}^0(t-\tau;E,\rho)\,\mathrm{d}\tau=\int_0^t u(\tau)\,\frac{\partial^2 K_{nm}^0(t-\tau;E,\rho)}{\partial t^2}\mathrm{d}\tau=$$

$$=-\frac{1}{\rho}\sqrt{\frac{E}{\rho}}\cdot\lambda_{nm}^2\int_0^t u(\tau)\sin\left(\sqrt{\frac{E}{\rho}}\cdot\lambda_{nm}(t-\tau)\right)\mathrm{d}\tau.$$

Therefore,

$$\frac{\partial^2\Delta w_0}{\partial t^2}=\sum_{n,m=1}^{\infty}v_{nm}^0\Delta\left(\varphi_n(x)\,\varphi_m(y)\right)\int_0^t u(\tau)\,\frac{\partial^2 K_{nm}^0(t-\tau;E,\rho)}{\partial t^2}\mathrm{d}\tau=$$

$$=\frac{\pi^2}{l^2}\frac{1}{\rho}\sqrt{\frac{E}{\rho}}\sum_{n,m=1}^{\infty}\lambda_{nm}^2\left(n^2+m^2\right)v_{nm}^0\varphi_n(x)\,\varphi_m(y)\times$$

$$\times\int_0^t u(\tau)\sin\left(\sqrt{\frac{E}{\rho}}\cdot\lambda_{nm}(t-\tau)\right)\mathrm{d}\tau.$$

Substituting this expression into w_2, we will obtain

$$w_2(x,y,t;\boldsymbol{P})=\frac{4\pi^2h^2}{10l^4\left(1-\nu^2\right)}\frac{E}{\rho G'}\int_0^t\int_0^l\int_0^l\sum_{n,m=1}^{\infty}\lambda_{nm}^2\left(n^2+m^2\right)v_{nm}^0\times$$

$$\times\varphi_n(\xi)\,\varphi_m(\eta)\int_0^\tau u(\tau)\sin\left(\sqrt{\frac{E}{\rho}}\cdot\lambda_{nm}(\tau-\tau_1)\right)\mathrm{d}\tau_1\times$$

$$\times\sum_{n,m=1}^{\infty}\frac{1}{\lambda_{nm}}\sin\left(\sqrt{\frac{E}{\rho}}\cdot\lambda_{nm}(t-\tau)\right)\varphi_n(x)\,\varphi_n(\xi)\,\varphi_m(y)\,\varphi_m(\eta)\,\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\tau.$$

On the other hand, since

$$\int_0^l\varphi_{n_1}(\xi)\,\varphi_{n_2}(\xi)\,\mathrm{d}\xi=\frac{l}{2}\delta_{n_1}^{n_2},$$

where $\delta_{n_1}^{n_2}$ is the Kronecker symbol, we will obtain

$$w_2(x,y,t;\boldsymbol{P})=\sum_{n,m=1}^{\infty}v_{nm}^2\varphi_n(x)\,\varphi_m(y)\int_0^t K_{nm}^2(u,t,\tau;\boldsymbol{P})\,\mathrm{d}\tau,$$

where

$$v_{nm}^2=\frac{\pi^2h^2}{10l^2\left(1-\nu^2\right)}\cdot\lambda_{nm}\left(n^2+m^2\right)v_{nm}^0.$$

$$K_{nm}^2(u,t,\tau;\boldsymbol{P})=\frac{E}{\rho G'}\cdot\sin\left(\sqrt{\frac{E}{\rho}}\cdot\lambda_{nm}(t-\tau)\right)\times$$

$$\times\int_0^\tau u(\tau_1)\sin\left(\sqrt{\frac{E}{\rho}}\cdot\lambda_{nm}(\tau-\tau_1)\right)\mathrm{d}\tau_1.$$

Thus, finally,

$$
\begin{aligned}
w\left(x, y, t ; \boldsymbol{P}\right)=\sum_{n, m=1}^{\infty}\Bigg[& v_{nm}^{0} \int_{0}^{t} u(\tau) K_{nm}^{0}(t-\tau ; E, \rho) \,\mathrm{d} \tau- \\
& -v_{nm}^{1} \int_{0}^{t} u(\tau) K_{nm}^{1}(t-\tau ; \boldsymbol{P}) \,\mathrm{d} \tau+ \\
& +v_{nm}^{2} \int_{0}^{t} K_{nm}^{2}(u, t, \tau ; \boldsymbol{P}) \,\mathrm{d} \tau\Bigg] \varphi_{n}(x) \varphi_{m}(y) .
\end{aligned} \tag{2.5}
$$

3 Averaged controllability problem

Assume that the material parameters $\boldsymbol{P}=(\rho,\ E,\ G')$ are uniformly distributed random variables. Let the aim of control be to provide at $t=T$ the desired state

$$
w(x, y, T)=w_{T}(x, y), \quad \left.\frac{\partial w}{\partial t}\right|_{t=T}=\dot{w}_{T}(x, y) \quad \text{in} \quad \Omega
$$

with $w_{T}, \dot{w}_{T} \in L^{2}\left([0,1]^{2}\right)$. Then, the averaged controllability criterion will be

$$
\mathcal{R}_{T}(u)=\|\bar{w}(x, y, T)-w_{T}(x, y)\|_{L^{2}\left([0,1]^{2}\right)}^{2}+\left\|\left.\frac{\partial \bar{w}}{\partial t}\right|_{t=T}-\dot{w}_{T}(x, y)\right\|_{L^{2}\left([0,1]^{2}\right)}^{2} . \tag{3.1}
$$

3.1 Determination of averaged state

The averaged state $\bar{w}$ will be determined as

$$
\begin{aligned}
\bar{w}(x, y, t)=\int_{\mathbb{P}} w(x, y, t) p(\boldsymbol{P}) \,\mathrm{d} \boldsymbol{P}= & \int_{\mathbb{P}} w_{0}(x, y, t) p(\boldsymbol{P}) \,\mathrm{d} \boldsymbol{P}- \\
-\int_{\mathbb{P}} w_{1}(x, y, t) p(\boldsymbol{P}) \,\mathrm{d} \boldsymbol{P}+ & \int_{\mathbb{P}} w_{2}(x, y, t) p(\boldsymbol{P}) \,\mathrm{d} \boldsymbol{P}.
\end{aligned}
$$

Now, let us evaluate the averaged state taking into account that

$$
p(E)=\frac{\theta\left(E-E_{0}\right)-\theta\left(E-E_{1}\right)}{E_{1}-E_{0}},
$$

where θ is the Heaviside function. Since only K_{nm}^{0} depends on E and ρ only, we will obtain

$$
\bar{w}_{0}(x, y, t)=\frac{1}{\mu} \sum_{n, m=1}^{\infty} v_{nm}^{0} \varphi_{n}(x) \varphi_{m}(y) \cdot \int_{0}^{t} u(\tau) \bar{K}_{nm}^{0}(t-\tau) \,\mathrm{d} \tau,
$$

where $\mu=\left(E_{1}-E_{0}\right)\left(\rho_{1}-\rho_{0}\right)\left(G_{1}'-G_{0}'\right)$,

$$
\bar{K}_{nm}^{0}(t) \,\mathrm{d} \tau=\int_{\rho_{0}}^{\rho_{1}} \int_{E_{0}}^{E_{1}} K_{nm}^{0}(t ; E, \rho) \,\mathrm{d} E \mathrm{d} \rho.
$$

Since K^1_{nm} contains only $1/G'$, their integration is separate and straightforward. As a result, we obtain,

$$\bar{w}_1(x,y,t) = \frac{1}{\mu}\ln\frac{G'_1}{G'_0}\sum_{n,m=1}^{\infty} v^1_{nm}\varphi_n(x)\varphi_m(y)\cdot\int_0^t u(\tau)\bar{K}^1_{nm}(t-\tau)\,\mathrm{d}\tau,$$

$$\bar{K}^1_{nm}(t) = \int_{\rho_0}^{\rho_1}\int_{E_0}^{E_1}\sqrt{\frac{E}{\rho}}\sin\left(\sqrt{\frac{E}{\rho}}\cdot\lambda_{nm}t\right)\mathrm{d}E\,\mathrm{d}\rho.$$

Similarly,

$$\bar{w}_2(x,y,t) = \frac{1}{\mu}\ln\frac{G'_1}{G'_0}\sum_{n,m=1}^{\infty} v^2_{nm}\varphi_n(x)\varphi_m(y)\int_0^t \bar{K}^2_{nm}(u,t,\tau)\,\mathrm{d}\tau,$$

where

$$\begin{aligned}\bar{K}^2_{nm}(u,t,\tau) = \int_0^{\tau} u(\tau_1)\int_{\rho_0}^{\rho_1}\int_{E_0}^{E_1}\frac{E}{\rho}\cdot\sin\left(\sqrt{\frac{E}{\rho}}\cdot\lambda_{nm}(t-\tau)\right)\times\\ \times\sin\left(\sqrt{\frac{E}{\rho}}\cdot\lambda_{nm}(\tau-\tau_1)\right)\mathrm{d}E\,\mathrm{d}\rho\,\mathrm{d}\tau_1 :=\\ := \int_0^{\tau} u(\tau_1)K^3_{nm}(t,\tau,\tau_1)\,\mathrm{d}\tau_1.\end{aligned}$$

Thence, we arrive at

$$\bar{w}(x,y,t) = \sum_{n,m=1}^{\infty} K_{nm}(u,t)\varphi_n(x)\varphi_m(y), \tag{3.2}$$

where

$$\begin{aligned}K_{nm}(u,t) = \frac{v^0_{nm}}{\mu}\int_0^t u(\tau)\bar{K}^0_{nm}(t-\tau)\,\mathrm{d}\tau -\\ -\frac{1}{\mu}\ln\frac{G'_1}{G'_0}\int_0^t\Big[v^1_{nm}u(\tau)\bar{K}^1_{nm}(t-\tau) -\\ - v^2_{nm}\int_0^{\tau} u(\tau_1)K^3_{nm}(t,\tau,\tau_1)\,\mathrm{d}\tau_1\Big]\mathrm{d}\tau.\end{aligned}$$

3.2 Reduction of the averaged controllability condition to an infinite system

Denoting by $w_{T,nm}$ and $\dot{w}_{T,nm}$ the Fourier sine-coefficients of the corresponding functions, substituting (3.2) evaluated at $t = T$ into (3.1) and taking into account the orthogonality of trigonometric functions, (3.1) is reduced to the following coupled infinite system of constraints with respect to the control function u:

$$\begin{cases} K_{nm}(u,T) - w_{T,nm} = 0,\\ \left.\dfrac{\partial K_{nm}}{\partial t}\right|_{t=T} - \dot{w}_{T,nm} = 0.\end{cases} \tag{3.3}$$

4 Some heuristic solutions

Even though infinite system (3.3) is linear in u, its determination is not straightforward. For instance, it can be treated as a problem of moments and resolved explicitly (see [14] for details). It can also be formally satisfied by some heuristic solutions [15].

4.1 Trigonometric solution

Since $v_{nm}^{0,1,2}$, $w_{T,nm}$ and $\dot{w}_{T,nm}$ converge in n, m very fast, (3.3) can be truncated for some finite N, M. We additionally assume that u is compactly supported on $[0, T]$. Then,

$$u(t) = \sum_{k=1}^{L} u_k \sin(q_k t + r_k)$$

can be substituted into the truncated system to derive

$$\begin{cases} \displaystyle\sum_{k=1}^{L} u_k \mathcal{T}_{nm}(q_k, r_k) - w_{T,nm} = 0, \\ \displaystyle\sum_{k=1}^{L} u_k \left.\frac{\partial \mathcal{T}_{nm}}{\partial t}\right|_{t=T} - \dot{w}_{T,nm} = 0, \end{cases} \quad n \leq N, \quad m \leq M, \tag{4.1}$$

where

$$\begin{aligned} \mathcal{T}_{nm}(q_k, r_k) = \frac{v_{nm}^0}{\mu} \int_0^T \sin(q_k \tau + r_k) \bar{K}_{nm}^0 (T - \tau) \, \mathrm{d}\tau - \\ - \frac{1}{\mu} \ln \frac{G_1'}{G_0'} \int_0^T \Big[v_{nm}^1 \sin(q_k \tau + r_k) \bar{K}_{nm}^1 (T - \tau) - \\ - v_{nm}^2 \int_0^\tau \sin(q_k \tau_1 + r_k) K_{nm}^3 (T, \tau, \tau_1) \, \mathrm{d}\tau_1 \Big] \mathrm{d}\tau. \end{aligned}$$

4.2 Piecewise constant solution

Another important control class is represented as

$$u(t) = \sum_{k=1}^{L} u_k \left[\theta(t - t_{k-1}) - \theta(t - t_k) \right].$$

In this case, the parameters u_k, t_k will be determined from the following system:

$$\begin{cases} \displaystyle\sum_{k=1}^{L} u_k \mathcal{P}_{nm}(t_k) - w_{T,nm} = 0, \\ \displaystyle\sum_{k=1}^{L} u_k \left.\frac{\partial \mathcal{P}_{nm}}{\partial t}\right|_{t=T} - \dot{w}_{T,nm} = 0, \end{cases} \quad n \leq N, \quad m \leq M, \tag{4.2}$$

where

$$\mathcal{P}_{nm}(t_k) = \frac{v^0_{nm}}{\mu} \int_{t_{k-1}}^{t_k} \bar{K}^0_{nm}(T-\tau)\,\mathrm{d}\tau - \frac{1}{\mu} \ln \frac{G'_1}{G'_0} \left[v^1_{nm} \int_{t_{k-1}}^{t_k} \bar{K}^1_{nm}(T-\tau)\,\mathrm{d}\tau - \right.$$

$$\left. - v^2_{nm} \int_0^T \int_0^\tau \left[\theta(\tau_1 - t_{k-1}) - \theta(\tau_1 - t_k)\right] K^3_{nm}(T,\tau,\tau_1)\,\mathrm{d}\tau_1 \mathrm{d}\tau \right].$$

4.3 Impulsive control

Another heuristic solution is the impact action of the external force which can be represented in terms of Dirac function as follows:

$$u(t) = \sum_{k=1}^{L} u_k \delta(t - t_k).$$

In this case, for u_k and t_k we derive

$$\begin{cases} \displaystyle\sum_{k=1}^{L} u_k \mathcal{I}_{nm}(t_k) - w_{T,nm} = 0, \\ \displaystyle\sum_{k=1}^{L} u_k \left.\frac{\partial \mathcal{I}_{nm}}{\partial t}\right|_{t=T} - \dot{w}_{T,nm} = 0, \end{cases} \quad n \le N, \quad m \le M, \tag{4.3}$$

where

$$\mathcal{I}_{nm}(t_k) = \frac{v^0_{nm}}{\mu} \bar{K}^0_{nm}(T-t_k) - \frac{1}{\mu} \ln \frac{G'_1}{G'_0} \left[v^1_{nm} \bar{K}^1_{nm}(T-t_k) - \right.$$

$$\left. - v^2_{nm} \int_0^T K^3_{nm}(T,\tau,t_k)\,\mathrm{d}\tau \right].$$

Note that (4.1), (4.2) and (4.3) can be solved for appropriate L by efficient numerical methods of nonlinear programming, which will be the subject of another publication elsewhere.

5 Conclusions

Averaged controllability of Ambartsumyan plate made of a transversely isotropic material is studied. The plate, which is initially in complete equilibrium, is simply supported at its edges and is subject to an external control action with a prescribed distribution function on the upper surface of the plate. The material characteristics of the plate (more specifically, the density and both Young moduli) are considered to be uniformly distributed random variables and the averaged state of the plate is computed. The averaged controllability analysis of the plate is reduced to an infinite system of linear constraints with respect to the control function. Three distinct parametric families of heuristic controls are provided to satisfy the truncated version of the infinite system. Efficient numerical methods of nonlinear programming can be applied to determine those parameters.

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Received 18.03.2022