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SHEAR WAVE REFLECTION AND TRANSMISSION FROM SYMMETRICALLY AND ASYMMETRICALLY ARRANGED BI-MATERIAL LAYER

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Key words: shear wave, transmission ratio, reflection, propagator matrix

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Преломление и отражение сдвиговой волны от слоя с двумя симметрично и антисимметрично расположенными материалами

Ключевые слова: сдвиговая волна, коэффициент прохождения, отражение, матрица распространения

В данной работе изучается распространение сдвиговой волны в пространстве через слой с симметрично и асимметрично расположенными двумя материалами. Задача решена с помощью метода матрицы распространения и получено решение в общем виде. Для частичного случая коэффициенты передачи энергии рассчитаны численно и представлены на графиках.

Օհանյան Ս.Կ.

Սահքի ալիքների բեկումը և անդրադարձումը սիմետրիկ և հակասիմետրիկ դասավորված երկու նյութից կազմված շերտում

Հիմնաբառեր. Սահքի ալիք, ներթափանցման գործակից, անդրադարձում, տարածման մատրից

Հոդվածում ուսումնասիրվում է երկու կիսատարածությունները բաժանող սիմետրիկ և հակասիմետրիկ դասավորված շերտում սահքի ալիքի տարածումը: Դիտարկվող խնդիրը լուծվել է օգտագործելով տարածման մատրիցի մեթոդը, դուրս են բերվել լուծումների ընդհանրացված տեսքերը: Մասնավոր դեպքի համար կատարվել է թվային հաշվարկ և գրաֆիկորեն պատկերվել է ալիքի ներթափանցման էներգիայի գործակիցների կախվածությունը ընկնող ալիքի անկյունից:

In this paper shear wave propagation through a symmetrically and asymmetrically arranged bi-material layer sandwiched between two semi-spaces is studied. The problem was solved by propagator matrix method and solution in general form was obtained. For partial case transmission energy ratios are numerically calculated and graphically represented.

Introduction

Numerous problems of wave propagation in elastic multilayered medium were considered by Brekhovskikh [1]. In this paper special cases of shear wave reflection refraction problem in layered structure is studied. Electromagnetic reflection problems in stratified dielectric media is studied in [2]. SH wave reflection and transmission from elastic/piezoelectric and piezoelectric/piezoelectric interfaces are considered in [3]. Wave transmission through piezoelectric phononic crystal are considered in [4, 5]. Electroacoustic wave transmission from non-acoustic interface of piezoelectric materials of different symmetries is investigated in [6].

Statement of the problem

Bi-material layer composed of symmetrically and asymmetrically arranged elastic layers sandwiched between elastic semi-spaces are considered. The SH wave propagates from the lower $x \in (-\infty; 0]$ semi-space partially reflects at the interface of layered structure $x = 0$ and partially transmits to upper $x \in [d; \infty)$ semi-space. We consider two cases of layered structure sandwiched between two semi-spaces.

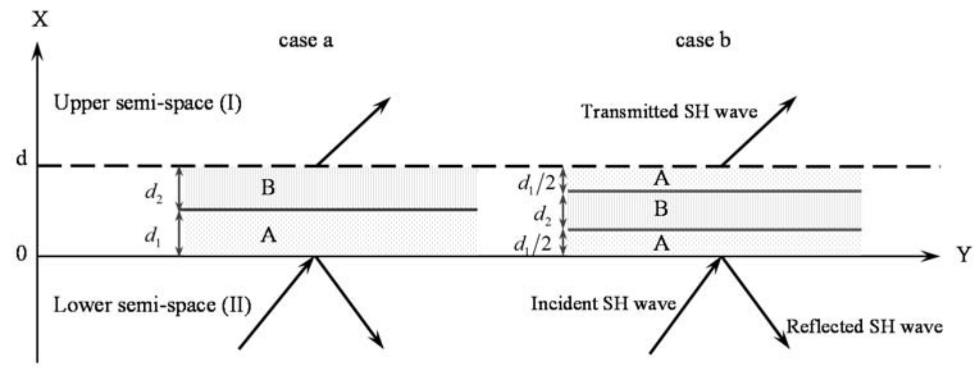

Fig. 1. Wave propagation in bi-material structure, asymmetrical layer (case a) and symmetrical layer (case b)

Layered structure in case *a)* is composed of two sub-layers with thicknesses $h_A = d_1$ and $h_B = d_2$, in case *b)* layer consists of three sub-layers with thicknesses $h_A = d_1/2,\ h_B = d_2$. Sub-layers are under ideal contact conditions with adjacent interfaces.

For anti-plane shear wave we have the following equation of motion

$$\partial_x \sigma_{xz} + \partial_y \sigma_{yz} = \rho\, \partial_{t,t} u_z \tag{1}$$

where stresses obey the Hooke's law

$$\sigma_{xz} = \mu\, \partial_x u_z,\ \ \sigma_{yz} = \mu\, \partial_y u_z \tag{2}$$

here $u_z(x, y, t)$ is the elastic displacement, μ, ρ are shear modulus and bulk density of material, respectively. For propagating SH wave the solution of equation of motion can be represented as

$$u_z(x, y, t) = u(x) \exp\left[i(k\, y - \omega t)\right] \tag{3}$$

where k, ω are wave number and frequency respectively.

Solution for functions $u^{(s)}(x)$ within $s = A, B$ materials of the layer can be found as

$$u^{(s)}(x) = \alpha_i^{(s)} \exp(iq_s x) + \alpha_r^{(s)} \exp(-iq_s x) \tag{4}$$

here $q_s = \sqrt{\omega^2/c_s^2 - k^2}$, $c_s^2 = \mu_s/\rho_s$, $s = A, B$. μ_s , ρ_s are shear moduli and bulk densities respectively, $\alpha_i^{(s)}, \alpha_r^{(s)}$ are complex amplitudes.

According to (2) one can define $\sigma_{zx}^{(s)}$ as

$$\sigma_{zx}^{(s)} = \tau^{(s)}(x) \exp\left[i(ky - \omega t)\right] \tag{5}$$

$$\tau^{(s)}(x) = i\mu_s q_s \left[\alpha_i^{(s)} \exp(iq_s x) - \alpha_r^{(s)} \exp(-iq_s x)\right] \tag{6}$$

At the interfaces of two materials the transmission conditions of the stress and displacement continuities can be imposed as

$$u^{(A)}(x) = u^{(B)}(x),\ \tau^{(A)}(x) = \tau^{(B)}(x),\quad x = h_A \quad \text{(case a)} \tag{7}$$

For case b) besides of condition (7) we have additional condition

$$u^{(B)}(x) = u^{(A)}(x),\ \tau^{(B)}(x) = \tau^{(A)}(x),\ x = h_A + h_B \tag{8}$$

Since the interface continuity conditions are imposed on functions $u^{(s)}(x)$, $\tau^{(s)}(x)$ it is convenient to introduce the following column vectors

$$\bar{U}^{(s)}(x) = \begin{pmatrix} u^{(s)} \\ \tau^{(s)} \end{pmatrix},\ \bar{A}^{(s)} = \begin{pmatrix} \alpha_i^{(s)} \\ \alpha_r^{(s)} \end{pmatrix} \tag{9}$$

In matrix form the solutions (5, 6) can be cast as

$$\bar{U}^{(s)}(x) = \hat{\mathrm{F}}^{(s)}(x) \cdot \bar{A}^{(s)} \tag{10}$$

where

$$\hat{\mathrm{F}}^{(s)}(x) = \begin{pmatrix} \exp(\mathrm{i}q_s x), & \exp(\text{-}\mathrm{i}q_s x) \\ \mathrm{i}\mu_s q_s \exp(\mathrm{i}q_s x), & -\mathrm{i}\mu_s q_s \exp(\text{-}\mathrm{i}q_s x) \end{pmatrix} \tag{11}$$

Let note that the transmission conditions (7,8) lead to the conditions of continuities of the field vectors $\bar{U}^{(s)}(x)$ at separation interfaces of the sub-layers.

Propagator matrix method

The solution of observed problem can be drawn by linking vectors $\bar{U}^{(A)}(x), \bar{U}^{(B)}(x)$ between top $x = d$ and bottom $x = 0$ surfaces of the layer with help of propagator matrix method [7]. According to this approach we need to consider first two points $x_1^{(s)}, x_2^{(s)}$ within each material in domains of the sub-layers $s = A, B$. For values of field vectors $\bar{U}^{(s)}(x)$ following conditions are valid

$$\bar{U}^{(s)}\left(x_1^{(s)}\right) = \hat{\mathrm{F}}^{(s)}\left(x_1^{(s)}\right) \cdot \bar{A}^{(s)},\quad \bar{U}^{(s)}\left(x_2^{(s)}\right) = \hat{\mathrm{F}}^{(s)}\left(x_2^{(s)}\right) \cdot \bar{A}^{(s)} \tag{12}$$

Eliminating vectors $\bar{A}^{(s)}$ from (12) the relation linking $\bar{U}^{(s)}$ vector field values within each material can be found as

$$\bar{U}^{(s)}\left(x_2^{(s)}\right)=T^{(s)}\left(x_1^{(s)},x_2^{(s)}\right)\bar{U}^{(s)}\left(x_1^{(s)}\right) \tag{13}$$

herein $\hat{T}^{(s)}\left(x_1^{(s)},x_2^{(s)}\right)=\hat{\mathrm{F}}^{(s)}\left(x_2^{(s)}\right)\left(\hat{\mathrm{F}}^{(s)}\left(x_1^{(s)}\right)\right)^{-1}$ is the transfer matrix in each sub-layer.

$$T^{(s)}\left(x_1^{(s)},x_2^{(s)}\right)=\begin{pmatrix} \cos\left(q_s\left(x_2^{(s)}-x_1^{(s)}\right)\right) & \left(\mu_s q_s\right)^{-1}\sin\left(q_s\left(x_2^{(s)}-x_1^{(s)}\right)\right) \\ -\mu_s q_s\sin\left(q_s\left(x_2^{(s)}-x_1^{(s)}\right)\right) & \cos\left(q_s\left(x_2^{(s)}-x_1^{(s)}\right)\right) \end{pmatrix} \tag{14}$$

Let now consider the layer with two sub-layers case of the structure.
In the case *a)* using the continuity conditions of field vectors $\bar{U}^{(s)}(x)$ at interface $x=d_1$ we come to the matrix equations

$$\bar{U}^{(A)}\left(d\right)=\hat{M}\bar{U}^{(B)}\left(0\right) \tag{15}$$

where

$$\hat{M}=\hat{T}^{(B)}\left(d_1,d_1+d_2\right)\hat{T}^{(A)}\left(0,d_1\right) \tag{16}$$

Herein $\hat{M}$ is the propagator matrix for shear wave field, which links the field vectors at the top and bottom of the layered structure.

$$\hat{M}=\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$

The explicit expressions of the unimodal propagator matrix $\hat{M}$ elements can be derived as

$$\begin{aligned} m_{11}&=\cos\left(d_1q_1\right)\cos\left(d_2q_2\right)-\frac{q_1\mu_1}{q_2\mu_2}\sin\left(d_1q_1\right)\sin\left(d_2q_2\right) \\ m_{12}&=\frac{\cos\left(d_1q_1\right)\sin\left(d_2q_2\right)}{q_2\mu_2}+\frac{\cos\left(d_2q_2\right)\sin\left(d_1q_1\right)}{q_1\mu_1} \\ m_{21}&=-q_2\mu_2\cos\left(d_1q_1\right)\sin\left(d_2q_2\right)-q_1\mu_1\cos\left(d_2q_2\right)\sin\left(d_1q_1\right) \\ m_{22}&=\cos\left(d_1q_1\right)\cos\left(d_2q_2\right)-\frac{q_2\mu_2}{q_1\mu_1}\sin\left(d_1q_1\right)\sin\left(d_2q_2\right) \end{aligned} \tag{17}$$

For the case *b)* the $\hat{M}$ matrix is constructed in a similar way

$$\hat{M}=\hat{T}^{(A)}\left(d_1/2+d_2,d\right)\hat{T}^{(B)}\left(d_1/2,d_1/2+d_2\right)\hat{T}^{(A)}\left(0,d_1/2\right)$$

matrix elements of which are

$$m_{11}=\cos\left(d_2q_2\right)\cos\left(d_1q_1\right)-\frac{\sin\left(d_2q_2\right)\sin\left(d_1q_1\right)\left(q_1^2\mu_1^2+q_2^2\mu_2^2\right)}{2q_1\mu_1q_2\mu_2},$$

$$m_{12}=\frac{\cos^2\left(\frac{1}{2}d_1q_1\right)\sin\left(d_2q_2\right)}{q_2\mu_2}+\frac{q_1\mu_1\cos\left(d_2q_2\right)\sin\left(d_1q_1\right)-q_2\mu_2\sin^2\left(d_2q_2\right)\sin\left(\frac{1}{2}d_1q_1\right)}{q_1^2\mu_1^2}$$

$$m_{21}=-q_1\mu_1\cos\left(d_2q_2\right)\sin\left(d_1q_1\right)-\frac{\left(q_1^2\mu_1^2\left(\cos\left(d_1q_1\right)-1\right)+q_2^2\mu_2^2\left(\cos\left(d_1q_1\right)+1\right)\right)\sin\left(d_2q_2\right)}{2q_2\mu_2} \tag{18}$$

$$m_{22}=\cos\left(d_2q_2\right)\cos\left(d_1q_1\right)-\frac{\sin\left(d_2q_2\right)\sin\left(d_1q_1\right)\left(q_1^2\mu_1^2+q_2^2\mu_2^2\right)}{2q_1\mu_1q_2\mu_2}$$

Reflection and transmission coefficients

For shear wave propagating in lower (I) and upper (II) semi-spaces we write solutions in the form

$$u_I(x,y,t)=u_I\left(x\right)\exp\left(i(py-\omega t)\right),\; u_{II}(x,y,t)=u_{II}\left(x\right)\exp\left(i(py-\omega t)\right)$$

$$u_I\left(x\right)=A_i\exp(iq_Ix)+A_r\exp(-iq_Ix),\; u_{II}\left(x\right)=A_t\exp(iq_{II}x)$$

$$\tau_I(x)=i\mu_Iq_I\left(A_i\exp(iq_Ix)-A_r\exp(-iq_Ix)\right)$$

$$\tau_{II}(x)=A_ti\mu_{II}q_{II}\exp(iq_{II}x)$$

where $q_s=\sqrt{\omega^2/c_s^2-k^2}$, $c_s^2=\mu_s/\rho_s$, $s=I,II$, A_i,A_r,A_t are amplitudes of incident, reflected, and refracted (transmitted) waves, respectively, μ_s are shear moduli and ρ_s are bulk densities of lower and upper semi-spaces respectively.

At the interfaces $x=0$, $x=d$ the conditions of stresses and displacements continuities are imposed

$$u_I\left(0\right)=u_1\left(0\right),\; \tau_I(0)=\tau_1(0)$$
$$u_{II}\left(d\right)=u_2\left(d\right),\; \tau_{II}(d)=\tau_2(d) \tag{19}$$

or in matrix form

$$\bar{U}^{(A)}\left(0\right)=\begin{pmatrix}u_I\left(0\right)\\ \tau_I\left(0\right)\end{pmatrix},\; \bar{U}^{(B)}\left(d\right)=\begin{pmatrix}u_{II}\left(d\right)\\ \tau_{II}\left(d\right)\end{pmatrix} \tag{20}$$

Taking into account the transmission condition (20) and the link relation (15), the amplitudes A_r, A_t via A_i can be found by solving the following matrix equation

$$\hat{M}\begin{pmatrix}A_i+A_r\\ i\mu_Iq_I\left(A_i-A_r\right)\end{pmatrix}=\begin{pmatrix}A_t\exp\left(iq_{II}d\right)\\ i\mu_{II}q_{II}A_t\exp\left(iq_{II}d\right)\end{pmatrix} \tag{21}$$

Solution of (21) can be found in the form

$$\mathrm{A}_r = \mathrm{A}_i \frac{\left(q_I \mu_I q_{II} \mu_{II} m_{12} + m_{21}\right) + i\left(m_{22} q_I \mu_I - q_{II} \mu_{II} m_{11}\right)}{\left(q_I \mu_I q_{II} \mu_{II} m_{12} - m_{21}\right) + i\left(m_{22} q_I \mu_I + q_{II} \mu_{II} m_{11}\right)}$$
$$\mathrm{A}_t = \frac{2\,\mathrm{A}_i\, q_I \mu_I \exp\left(-i q_{II} d\right)}{\left(m_{22} q_I \mu_I + q_{II} \mu_{II} m_{11}\right) - i\left(q_I \mu_I q_{II} \mu_{II} m_{12} - m_{21}\right)} \tag{22}$$

Energy flux conservation is then expressed via reflection and refraction amplitudes by the following algebraic identity

$$q_I \mu_I \left|A_r\right|^2 + q_{II} \mu_{II} \left|A_t\right|^2 = q_I \mu_I \left|A_i\right|^2 \tag{23}$$

Here

$$\left|A_r\right|^2 = A_i^2 \left(1 - \frac{4 q_I q_{II} \mu_I \mu_{II}}{\left(m_{21}\right)^2 + \left(m_{11}\right)^2 q_I^2 \mu_I^2 + 2 q_I q_{II} \mu_I \mu_{II} + q_{II}^2 \mu_{II}^2 \left(\left(m_{22}\right)^2 + \left(m_{12}\right)^2 q_I^2 \mu_I^2\right)}\right)$$
$$\left|A_t\right|^2 = \frac{4 A_i^2 q_I^2 \mu_I^2}{\left(m_{21}\right)^2 + \left(m_{11}\right)^2 q_I^2 \mu_I^2 + 2 q_I q_{II} \mu_I \mu_{II} + q_{II}^2 \mu_{II}^2 \left(\left(m_{22}\right)^2 + \left(m_{12}\right)^2 q_I^2 \mu_{II}^2\right)} \tag{24}$$

From equations (24) the transmission and reflection energy ratios coefficients can be written as

$$R = \frac{\left|A_t\right|^2}{\left|A_i\right|^2},\; T = \frac{\left|A_r\right|^2}{\left|A_i\right|^2} \tag{25}$$

Numerical calculation results

For numerical calculations we consider that materials of the upper and lower semi-spaces are the same. In this case we have that $R + T = 1$

We will consider the three layouts of material arrangements within layer which is represented in Table 1.

Table 1

Layout	Semi-spaces	Material A	Material B
(1)	Cu	Si	Au
(2)	Au	Cu	Si
(3)	Si	Au	Cu

The material properties are presented in Table 2.

Table 2

	$\rho\ (kg/m^3)$	$\mu\ (GPa)$	$c\ (m/s)$
Cu	8960	40	2113
Au	19300	27	5387
Si	2330	68	1183

For different values of the layer dimensionless thickness $h = kd$ on the Fig. 2-7 the energy transmission ratios $R(\theta)$ are presented in dependence of angles of incident wave $\theta = \arcsin(k c_0/\omega)$, where $c_0 = \sqrt{\mu_0/\rho_0}$ is velocity of the incident wave.

Relative thickness of sub-layers is denoted by $\delta = d_1 / d$ where $d = d_1 + d_2$ (see Fig.1).

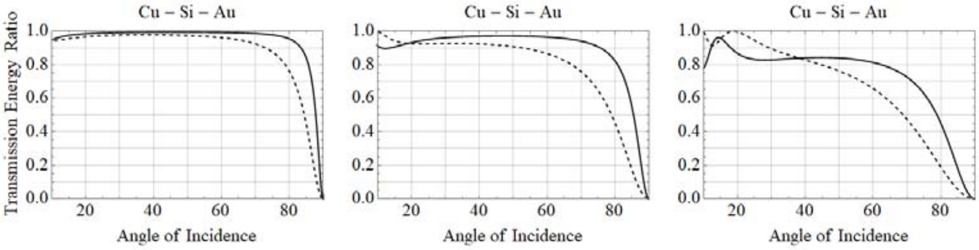

Fig. 2. Layout (1), $h = 0.2$ **,** $h = 0.4$ **,** $h = 0.8$ **,** $\delta = 0.4$ **for two layered (solid curve) and** $\delta = 0.2$ **for three layered cases (dashed curve)**

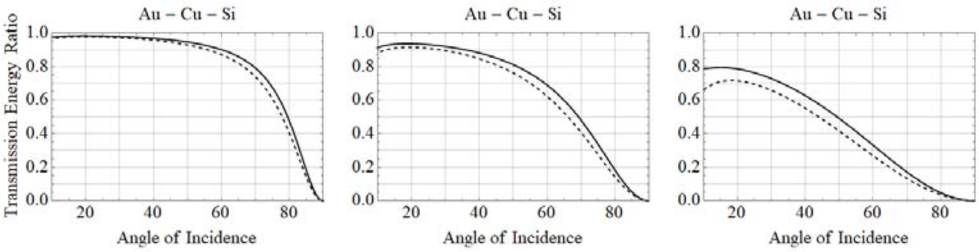

Fig. 3. Layout (2), $h = 0.2$ **,** $h = 0.4$ **,** $h = 0.8$ **,** $\delta = 0.4$ **for two layered (solid curve) and** $\delta = 0.2$ **for three layered cases (dashed curve)**

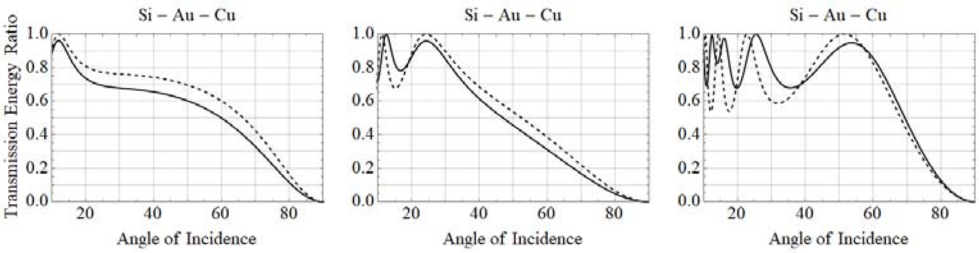

Fig. 4. Layout (3), $h = 0.2$ **,** $h = 0.4$ **,** $h = 0.8$ **,** $\delta = 0.4$ **for two layered (solid curve) and** $\delta = 0.2$ **for three layered cases (dashed curve)**

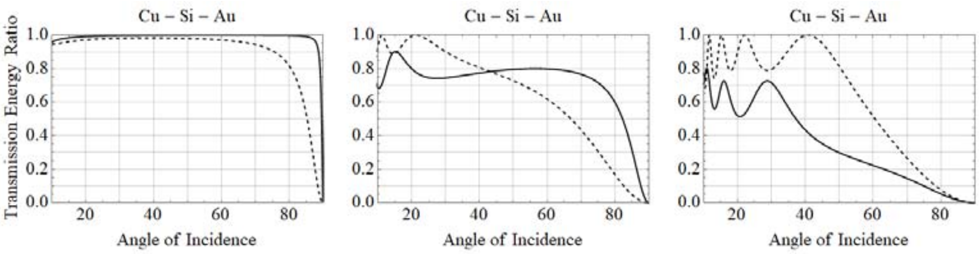

Fig. 5. Layout (1), $h = 0.2$ **,** $h = 1$ **,** $h = 2$ **,** $\delta = 0.5$ **for two layered (solid curve) and** $\delta = 0.25$ **for three layered cases (dashed curve)**

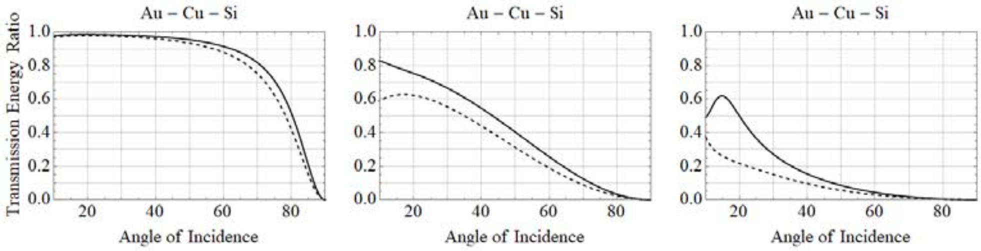

Fig. 6. Layout (2), $h = 0.2$, $h = 1$, $h = 2$, $\delta = 0.5$ **for two layered (solid curve) and** $\delta = 0.25$ **for three layered cases (dashed curve)**

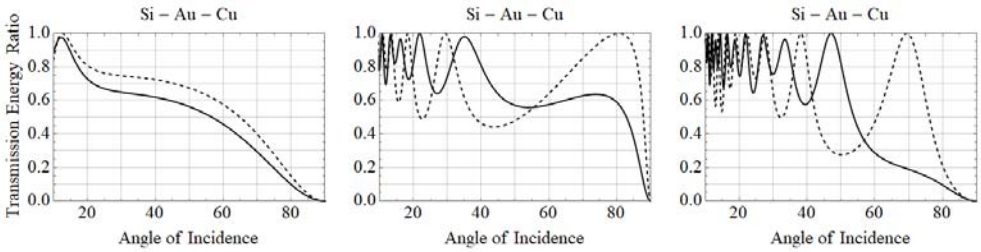

Fig. 7. Layout (3), $h = 0.2$, $h = 1$, $h = 2$, $\delta = 0.5$ **for two layered (solid curve) and** $\delta = 0.25$ **for three layered cases (dashed curve)**

As it follows from above data the values of energy transmission ratios qualitatively not considerably differ in the considered cases of symmetrically and asymmetrically arranged materials. By observation of differences between energy transmission ratios we can note that when wave propagate from hard semi-space (material Au) the values of transmission energy ratios are higher in asymmetrical case for all incident wave angles.

Conclusion

The problem of shear elastic wave propagation through a stratified bi- material layer was studied for asymmetrically and symmetrically arranged layers. By means of matrix propagator methods the amplitudes and energy ratios coefficients for incident and transmission waves are obtained analytically both for the symmetrical and asymmetrical layer. The numerical results and analysis are represented. For both cases the energy ratios are also plotted against the angle of incidence. Based on these results it is established that when the SH wave propagates from hard media the values of transmission energy ratios are higher in asymmetrical case comparing with symmetrical case.

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