ՀԱՅԱՍՏԱՆԻ ԳԻՏՈՒԹՅՈՒՆՆԵՐԻ ԱԶԳԱՅԻՆ ԱԿԱԴԵՄԻԱՅԻ ՏԵՂԵԿԱԳԻՐ
ИЗВЕСТИЯ НАЦИОНАЛЬНОЙ АКАДЕМИИ НАУК АРМЕНИИ

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ON THE STRESS-STRAIN STATE OF AN ELASTIC INFINITE PLATE WITH A CRACK EXPANDING BY MEANS OF SMOOTH THIN INCLUSION INDENTATION

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Keywords: elastic plate, crack, inclusion, cracks gap, stress intensity factor (SIF)

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О напряжённо-деформационном состоянии упругой бесконечной пластины с трещиной, расширяющейся посредством вдавливания в неё гладкого тонкого включения

Ключевые слова: упругая пластина, трещина, включение, раскрытие трещины, коэффициент интенсивности напряжений (КИН).

Рассматривается плоская задача об определении компонент напряжённо-деформированного состояния упругой изотропной бесконечной пластины с прямолинейной трещиной конечной длины, в которой вставлено тонкое абсолютно жёсткое включение с гладкой поверхностью. Предполагается, что эта поверхность обладает центральной и осевой симметрией, имеет форму типа сплюснутого эллипса или форму тонкого стержня прямоугольного сечения и при вдавливании плотно прилегает к берегам трещины, а на образовавшемся при этом контактном участке действуют только нормальные контактные напряжения. Решение обсуждаемой задачи сведено к решению интегрального уравнения Фредгольма первого рода с симметрическим логарифмическим ядром. Построены точное и численно-аналитическое приближённое решения этого уравнения. Рассмотрены частные случаи, проведён их численный анализ и исследованы закономерности изменения характеристик задачи.

Մխիթարյան Ս.Մ.

Ճաքով առաձգական անվերջ սալի լարվածադեֆորմացիոն վիճակի մասին, երբ ճաքն ընդլայնվում է նրան սեղմվող ողորկ բարակապատ ներդրակով

Հիմնաբառեր: առաձգական սալ, ճաք, ներդրակ, ճաքի բացվածք, լարումների ուժգնության գործակից:

Դիտարկվում է վերջավոր երկարության ուղղագիծ ճաքով առաձգական իզոտրոպ սալի լարվածադեֆորմացիոն վիճակի բաղադրիչների որոշման հարթ խնդիրը, երբ ճաքի մեջ ներդրված է ողորկ մակերևույթով բացարձակ կոշտ բարակապատ ներդրակ: Ենթադրվում է, որ այդ մակերևույթը օժտված է կենտրոնական և առանցքային համաչափությամբ և ունի տափակեցրած էլիպսի կամ ուղղագիծ բարակապատ ձողի տեսք և սեղմվելիս քիփ հպվում է ճաքի ափերին, իսկ առաջացած կոնտակտային տեղամասում գործում են միայն նորմալ կոնտակտային լարումներ: Խնդրի լուծումը բերված է սիմետրիկ լոգարիթմական կորիզով Ֆրեդհոլմի առաջին սեռի ինտեգրալ հավասարման լուծման: Կառուցված են այդ հավասարման փակ և մոտավոր թվային-վերլուծական լուծումներ: Դիտարկված են մասնավոր դեպքեր և կատարված են թվային վերլուծություններ, հետազոտված են խնդրի բնութագրիչների փոփոխման օրինաչափությունները:

The plane problem of determining components of the stress-strain state of an elastic isotropic infinite plate with a rectilinear crack of finite length, in which a thin rigid inclusion with a smooth surface is indented, is considered. We assumed that the inclusion surface has a centrally-axial symmetry, its shape is an oblate ellipse part or a thin rectangle, and during the indentation, the inclusion surface is tightly adjoined to edges of the crack, and only normal stresses act in the formed contact region. Solving the problem is reduced to solving the Fredholm integral equation of the first kind with a logarithmic symmetric kernel. The exact analytical and approximate numerical-analytical solutions of this equation convenient for engineering calculations are constructed. The main characteristics of the problem (such as contact normal stresses in the region of contact between inclusion and crack edges, the cracks gap (opening) outside the inclusion, normal breaking stresses outside the crack on its location line, their stress intensity factors (SIF)) are represented by explicit analytical formulas. In particular cases, the numerical analysis of these characteristics is carried out, regularities of their changes are revealed. The phenomenon of infinite increase of SIF is established at infinite approach of the end points of inclusion to the crack tips. As a result, the crack begins to propagate and brittle fracture of the plate occurs. Proceeding from this, as an application of the obtained results, an estimation of crack resistance of a plate with a crack is given; namely, the critical value of the relative length of the rectilinear inclusion, at which the crack propagation begins, is determined.

1. Introduction. Cracks and foreign inclusions in deformable solid bodies are stress concentrators, around which the local stress fields characterized by large and rapidly changing gradients are formed. The stress concentrations have a significant influence on the strength characteristics of structures. Therefore, the qualitative and quantitative investigation of the stress concentrations as well as the development of ways to reduce them have theoretical and practical significance. Problems of determining the stress strain state of deformable bodies with cracks and inclusions, issues regarding the interaction of cracks and inclusions and their influence on strength properties of structures are often encountered in the mechanics of composites, in geomechanics, in thermoelasticity, and in their engineering applications. These problems are the subject of numerous studies [1-7]. In [3] the problem on the development of a crack in the vicinity of the rigid inclusion top was studied, and SIFs near the crack, on the continuation of the linear rigid inclusion, are determined. Different cases of combination of cracks and rigid inclusions in an elastic matrix were investigated in [8]. Many results on this topic are summarized in the handbooks of SIFs [9, 10].

In the present paper, we consider the plane problem of determining components of stresses and displacements (playing an important role in fracture mechanics) of an elastic isotropic infinite plate with a rectilinear crack of finite length, in which a thin absolutely rigid inclusion with a smooth surface is indented. We assume that the surface has a centrally-axial symmetry, its shape is a part of an ellipse oblate along its length or a thin rectangle. During the indentation, the inclusion surface is tightly adjoined to the crack edges and only normal stresses act on formed contact regions. Solving the problem is reduced to solving the Fredholm integral equation of the first kind with the logarithmic symmetric kernel; its exact solution is obtained (the necessary formulas and transformations for construction of the exact analytical solution are transferred to Appendices A, B, C). Based on the Gauss-type quadrature formulas (like in [11]) in combination with the method of collocation, the approximate numerical-analytical solution of the governing equation is also obtained.

It should be noted that the problem discussed here represents a flat analog of the axisymmetric problem, previously considered in [7], where only SIF is approximately calculated using the model based on the solution of the classical contact problem on the indentation of a round punch with the flat base into the elastic half-space. Such model is applicable only for small relative lengths of inclusion. However, while investigating the problem of crack propagation and clarifying the issue of the maximal inclusion length which an elastic matrix with a crack can withstand, it is necessary to consider exactly the large values of the relative length of the inclusion. On the other hand, these problems are closely related to the problems of wedging elastic bodies by thin, absolutely rigid wedges of various shapes, widely covered in the handbooks [9, 10]. On this concern let us point out also the works close to our subject [12-16]. However, the problem formulations are different: in the problems on wedging elastic bodies, the positions of the end points of cracks and contact zones are unknown in advance and are determined in the course of problem-solving. In our case, these parameters are given in advance.

The main characteristics of the problem mentioned above are represented by explicit analytical formulas. In particular cases, regularities of change in these characteristics depending on the specific parameters are revealed by the numerical analysis.

As an application of the obtained results, an estimation of crack resistance of a plate with a crack is given for the rectilinear inclusion. For this case, the critical value of the relative length of the inclusion, at which the crack propagates occurs, is determined. Thus, a plate with a crack cannot withstand inclusion of any length; especially the inclusion with a length equal to the length of the crack, i.e. when there is a complete contact of the inclusion with the crack edges.

2. Problem formulation. Assume that an infinite elastic plate with the elastic modulus E, the Poisson's ratio ν and the thickness h in a rectangular coordinate system Oxy has a crack along the abscissa axis (mathematical cut) of finite length $2l$, $L=\{y=0,\ -l\le x\le l\}$ (Fig. 1). It is thought that the plate is isotropic, homogeneous, and is in the generalized plane stress state. Suppose that a thin absolutely rigid inclusion of length $2a\ (a<l)$ with a smooth surface is indented into the crack. The surface is described by equations $y=\pm f(x)\ (-a\le x\le a)$, where $f(x)$ is an even nonnegative function $(f(-x)=f(x))$; both the function $f(x)$ and its first derivative are continuous on the interval $[-a,a]$ and $\Delta=f(0)=\max\limits_{|x|\le a} f(x) << a(a<l)$.

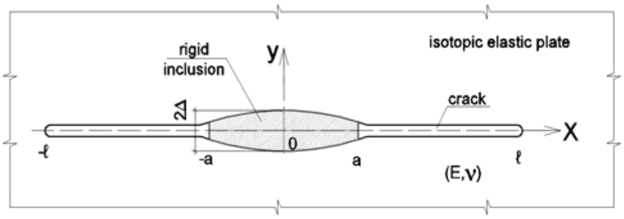

Fig.1. Smooth absolutely rigid oblate along its length inclusion with an upper surface $y=f(x)$ and lower surface $y=-f(x)$ putting pressure on crack edges $L=\{y=0,-l\le x\le l\}$ in an elastic infinite plate.

In particular, the thin inclusion may be in the form of a strongly oblate along its length ellipse part

$$\frac{x^2}{a_1^{\ 2}}+\frac{y^2}{b_1^{\ 2}}=1 \quad (b_1<<a, -a\le x\le a,\ a<a_1<l).$$

The minor semi-axis b_1 of the ellipse is much shorter than the major semi-axis a_1. Whence

$$f(x)=\frac{b_1}{a_1}\sqrt{a_1^{\ 2}-x^2}\ (-a\le x\le a;\ a<a_1\le l). \tag{1}$$

Such a choice of the shape of the inclusion is due to the fact that according to [17–19] the crack is considered as the limiting case of an ellipse when $b_1\to 0\ (a_1=l)$; in this sense geometric forms of the thin inclusion and crack are compatible. In addition, under this limiting transition, the stress state of an elastic infinite plate with a thin elliptic hole becomes a stress state of the plate caused by a Griffiths crack [19] (pp. 308-309).

Besides the described form, a thin inclusion can also be in the form of a rectilinear segment with length $2a$ and height 2δ, where $\delta << a$ (Fig.2).

Hereafter we consider the possibility of approaching the inclusion endpoints to the crack tips $(a\to l)$. In accordance with the crack interpretation as a limiting case of the ellipse (1) when $b_1\to 0$ [17–19], and taking into account well-known asymptotic formulas for displacements near the crack tips, it will be assumed that the function $f(x)$ has the following behavior

$f(x) = 0\left((l \pm x)^{-1/2}\right) \quad (x \to \mp l)$.

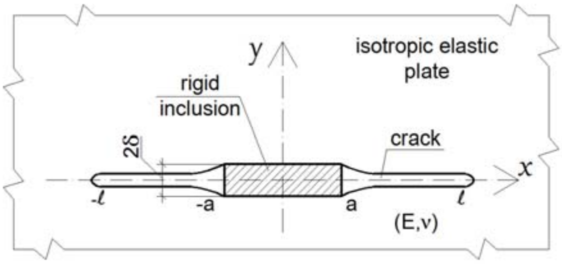

Fig.2. Smooth absolutely rigid linear inclusion with length $2a$ and height 2δ $(\delta << a,\ a < l)$ putting pressure on crack edges $L = \{y = 0,\ -l \le x \le l\}$ in an elastic infinite plate.

Then we suppose that the thin inclusion with a smooth surface, indenting into the crack edges, tightly adjoins them along its entire length, i.e. contact region is the line segment $-a \le x \le a$. Because of the smoothness of the inclusion surface, we assume that only normal stresses arise in the contact region.

Under these assumptions, it is required to determine the pressure of the inclusion surface on the crack edges or normal contact stresses, crack opening outside the inclusion, normal breaking stresses outside the crack along the line of its location and SIFs.

Due to symmetry about the x-axis, within a well-known approximation [20, pp. 114-115], according to which the boundary conditions from the walls of the inclusion can be transferred to its midline, the posed problem can be formulated as the following mixed boundary-value problem of the mathematical theory of elasticity for the elastic upper half-plane $y > 0$:

$$\left.\begin{aligned} &\mathrm{v}(x,y)\big|_{y=+0} = f(x)\ \left(-a \le x \le a,\ f(-x) = f(x) \ge 0\right),\quad \mathrm{v}(x,y)\big|_{y=+0} = 0\ \left(|x| \ge l\right),\\ &\tau_{xy}\big|_{y=+0} = 0 \quad (-\infty < x < \infty),\quad \sigma_y(x,y)\big|_{y=+0} = 0 \qquad \left(a < |x| < l\right),\\ &\sigma_x(x,y),\ \sigma_y(x,y)\ \tau_{xy}(x,y) \to 0 \quad \text{as} \quad x^2 + y^2 \to \infty. \end{aligned}\right\} \quad \text{(2a-e)}$$

Here $\mathrm{v}(x,y)$ is a vertical displacement of the point $M(x,y)$ of the upper elastic half-plane and σ_x, σ_y, τ_{xy} are components of normal and tangential stresses, respectively.

Reduce solving the problem (2a-e) to solving an integral equation. For this purpose, we use the solution of the auxiliary problem from Appendix A. Namely, with the help of (A7) fulfilling the boundary condition (2a), we arrive at the following Fredholm governing integral equation (IE) of the first kind with a symmetric kernel for unknown pressure $p(x)\ \left(p(-x) = p(x)\right)$ of the rigid inclusion on the crack edges:

$$\frac{1}{\pi E}\int_{-a}^{a} \ln\frac{l^2 - xs + \sqrt{\left(l^2 - x^2\right)\left(l^2 - s^2\right)}}{l^2 - xs - \sqrt{\left(l^2 - x^2\right)\left(l^2 - s^2\right)}}\, p(s)\,ds = f(x) \quad \left(|x| < a\right) \tag{3}$$

Further, in (3), (A8) – (10) we proceed to dimensionless coordinates and values, assuming

$$\xi = x/a,\ \ \eta = s/a,\ \ p_0(\xi) = p(a\xi)/E,\quad f_0(\xi) = f(a\xi)/a,$$
$$\delta_0 = \delta/a,\quad \rho = l/a\quad (\rho > 1),\quad \Psi_0(\xi) = \Psi(a\xi)/a, \tag{4}$$
$$K_I^0 = \sqrt{\pi l}K_I/aE,\quad \sigma_0(\xi) = \sigma(a\xi)/E,\quad \sigma_y^0(\xi) = \sigma_y(a\xi, +0)/E.$$

As a result, these formulas are transformed to the followings:

$$\frac{1}{\pi}\int_{-1}^{1}\ln\frac{\rho^2 - \xi\eta + \sqrt{(\rho^2-\xi^2)(\rho^2-\eta^2)}}{\rho^2 - \xi\eta - \sqrt{(\rho^2-\xi^2)(\rho^2-\eta^2)}}p_0(\eta)\,d\eta = f_0(\xi)\quad (-1<\xi<1); \tag{5}$$

$$\sigma_y^0(\xi) = -\sigma_0(\xi) = -\frac{1}{\pi\sqrt{\xi^2-\rho^2}}\int_{-1}^{1}\frac{\sqrt{\rho^2-\eta^2}\,p_0(\eta)\,d\eta}{\eta-\xi}\quad (\xi>\rho); \tag{6}$$

$$K_I = \left(aE/\sqrt{\pi l}\right)K_I^0;\ K_I^0 = \int_{-1}^{1}\sqrt{\frac{\rho+\eta}{\rho-\eta}}\,p_0(\eta)\,d\eta; \tag{7}$$

$$\Psi_0(\xi) = \frac{2}{\pi}\int_{-1}^{1}\ln\frac{\rho^2 - \xi\eta + \sqrt{(\rho^2-\xi^2)(\rho^2-\eta^2)}}{\rho^2 - \xi\eta - \sqrt{(\rho^2-\xi^2)(\rho^2-\eta^2)}}p_0(\eta)\,d\eta\quad (1\le\xi\le\rho). \tag{8}$$

The formulas (5)–(8) are the basic equations and relationships of the posed problem.

3. Method of analytical solution. We proceed to the solution of the governing IE (5) and first transform it to a simpler trigonometric form, assuming that

$$\xi = \rho\cos\vartheta,\quad \eta = \rho\cos\varphi;\quad \alpha = \arccos(1/\rho) = \arccos(a/l);$$
$$\omega_0(\vartheta) = p_0(\rho\cos\vartheta)\sin\vartheta;\ g_0(\vartheta) = f_0(\rho\cos\vartheta); \tag{9}$$
$$\alpha<\vartheta,\ \varphi<\beta;\ \beta = \pi-\alpha\ (0<\alpha<\pi/2).$$

As a result, after simple manipulations the equation (5) gets the following form:

$$\frac{2\rho}{\pi}\int_{\alpha}^{\beta}\ln\left[\sin\frac{(\vartheta+\varphi)}{2}\bigg/\left|\sin\frac{(\vartheta-\varphi)}{2}\right|\right]\omega_0(\varphi)\,d\varphi = g_0(\vartheta)\quad (\alpha<\vartheta<\beta). \tag{10}$$

Represent the solution of the integral equation (10) equivalent to the original equation (5) in the form of an infinite series with unknown coefficients $x_n\ (n = 0,1,2,...)$ as follows (see Appendix B):

$$\omega_0(\vartheta) = \frac{1}{\sqrt{\cos 2\alpha - \cos 2\vartheta}}\sum_{n=0}^{\infty}x_n T_n(\mathrm{X})\quad (\alpha<\vartheta<\pi-\alpha) \tag{11}$$

where X is given in (B9), $T_n(\mathrm{X})$ are the Chebyshev polynomials of the first kind. Further, we substitute (11) into (10), change the order of integration and summation, and use spectral relationships (B9). As a result, we get

$$2\rho\sum_{n=0}^{\infty}\mu_n x_n T_n(\mathrm{X}) = g_0(\vartheta) \qquad (\alpha < \vartheta < \pi - \alpha).$$

Whence using the orthogonality conditions (B11), we find

$$x_n = \frac{\sqrt{2}\cos^2\frac{\alpha}{2}\,h_n}{\rho K' k_n \mu_n};\ h_n = \int_{\alpha}^{\pi-\alpha}\frac{g_0(\vartheta)T_n(\mathrm{X})\,d\vartheta}{\sqrt{\cos 2\alpha - \cos 2\vartheta}};\ k_n = \begin{cases} 2\ (n=0); \\ 1\ (n \neq 0); \end{cases} (n = 0,1,2,...) \quad (12)$$

where μ_n is given by the formula (B9). Thus, the coefficients x_n are determined by formulas (12).

To solve the integral equation (5) for $\rho = 1$ $(a = l)$, according to (B19), we represent its solution as an infinite series with unknown coefficients y_n $(n = 1,2,...)$

$$p_0(\xi) = \sum_{n=1}^{\infty} y_n U_{n-1}(\xi) \qquad (-1 < \xi < 1), \tag{13}$$

where $U_{n-1}(\xi)$ are Chebyshev polynomials of the second kind.

Further, as above, we substitute (13) into equation (5) for $\rho = 1$, change the order of integration and summation, and then use the relations (B19). As a result, we obtain the equality

$$\sqrt{1-\xi^2}\sum_{n=1}^{\infty} n^{-1} y_n U_{n-1}(\xi) = 2^{-1} f_0(\xi) \qquad (-1 < \xi < 1).$$

Now multiply both sides of this equality by $U_{m-1}(\xi)$ $(m = 1,2,...)$ and integrate with respect of ξ from -1 to 1. Using the orthogonality condition of the Chebyshev polynomials of the second kind, we find

$$y_n = 2^{-1} n g_n;\quad g_n = 2\pi^{-1}\int_{-1}^{1} f_0(\xi) U_{n-1}(\xi)\,d\xi \quad (n = 1,2,...). \tag{14}$$

If we assume that the function $f_0(\xi)$ is a twice continuously differentiable function on the interval $(-1,1)$ and we take into account that $f_0(\pm 1) = 0$, then after integration by parts it becomes possible to get the estimation of the Fourier coefficients:

$g_n = O\left(1/n^{2+\varepsilon}\right)$ $(\varepsilon > 0)$ for $n \to \infty$

and, consequently, the series (13) or corresponding Fourier sine series converges uniformly on any interval $[-r, r]$ $(r < 1)$.

4. Solution of IE (5) by numerical-analytical method. We will also construct approximate solutions of the discussed integral equations using the numerical-analytical method based on the Gaussian quadrature formula for calculating definite integrals in conjunction with the collocation method. The same approach is in the basis of the well-known numerical-analytical method for solving singular integral equations (SIE) proposed in [21] and [22]. Based on these considerations the solution of equation (5) can be represented in the form

$$p_0(\xi)=\Omega(\xi,\rho)\big/\sqrt{1-\xi^2}\quad(-1<\xi<1),$$

where $\Omega(\xi,\rho)$ is the Hölder function in the interval $-1\le\xi\le1$. Further, following the well-known procedure, solving the equation (5) is reduced to solving the following finite set of linear algebraic equations

$$\left.\begin{array}{l}\displaystyle\sum_{m=1}^{N}K_{rm}\mathrm{X}_m=a_r\quad\left(r=\overline{1,N}\right)\\ \displaystyle K_{rm}=\frac{1}{N}\ln\frac{\rho^2-\xi_r\eta_m+\sqrt{\left(\rho^2-\xi_r^2\right)\left(\rho^2-\eta_m^2\right)}}{\rho^2-\xi_r\eta_m-\sqrt{\left(\rho^2-\xi_r^2\right)\left(\rho^2-\eta_m^2\right)}},\quad \mathrm{X}_m=\Omega(\eta_m,\rho),\quad a_r=f_0(\xi_r),\end{array}\right\}$$

$$\eta_m=\cos\left[(2m-1)\pi/2N\right],\ \ \xi_r=\cos\left[\pi r/(N+1)\right]\ \ \left(m,r=\overline{1,N}\right).\tag{15}$$

Here η_m and ξ_r are Chebyshev knots, roots of Chebyshev polynomials of the first kind $T_N(\eta)$ and the second kind $U_N(\xi)$, respectively, where N is any natural number. After solving the system of equations (15), the solution of (5) at Chebyshev knots, η_m, will be determined by the formula

$$p_0(\eta_m)=\mathrm{X}_m\big/\sqrt{1-\eta_m^2}\quad\left(m=\overline{1,N}\right).\tag{16}$$

In a similar manner, the solution of the same equation (5) for $\rho=1$ reduces to that of the system of equations (15), in which, however, one should put

$$\rho=1;\quad \mathrm{X}_m=\sqrt{1-\eta_m^2}\,\Omega(\eta_m,\rho)\ \ \left(m=\overline{1,N}\right).$$

5. Analytical results. Let us calculate the basic mechanical characteristics of the problem under consideration in the explicit analytical form. Turn first to the formula (11), from which the dimensionless pressure of the rigid inclusion on the crack edges is determined. In this formula, we return to the former variables and quantities. From (4) and (9) we get

$$x=a\xi,\quad \xi=\rho\cos\vartheta,\quad (\rho=l/a),\ \text{i.e. } x=l\cos\vartheta\ \ (\alpha<\vartheta<\pi-\alpha).$$

In the light of the above, we transform

$$\sqrt{\cos2\alpha-\cos2\vartheta}=\sqrt{1+\cos2\alpha-(1+\cos2\vartheta)}=\sqrt{2}\sqrt{\cos^2\alpha-\cos^2\vartheta}=$$

$$=\frac{1}{\rho\sqrt{2}}\sqrt{1-\cos^2\alpha}\,(\rho+\xi)\sqrt{\left(\frac{\rho+1}{\rho-1}-\frac{\rho-\xi}{\rho+\xi}\right)\left(\frac{\rho-\xi}{\rho+\xi}-\frac{\rho-1}{\rho+1}\right)}=\frac{2\sqrt{a^2-x^2}}{l\sqrt{2}}$$

$$(-a<x<a).$$

On the other hand, in formula (B9) assuming that

$$\vartheta=\arccos\frac{x}{l},\ \ t=\arccos\frac{s}{l}\qquad(-a<x,s<a),$$

after simple transformations we get

$$\mathrm{X}=\cos\Theta,\quad \Theta=\frac{\pi}{2K'}(a+l)\int\limits_x^a \frac{ds}{\sqrt{(a^2-s^2)(l^2-s^2)}}\qquad (-a<x<a). \tag{17}$$

Then from (9)

$$\omega_0(\vartheta)=p_0(\rho\cos\vartheta)\sin\vartheta=p_0(\xi)\sqrt{1-\xi^2/\rho^2}=p(x)\sqrt{l^2-x^2}\Big/El.$$

Taking into consideration these transformations, the formula (11) for the pressure of the inclusion on the crack edges is represented as

$$p(x)=\frac{El^2}{\sqrt{2}}\frac{1}{\sqrt{(a^2-x^2)(l^2-x^2)}}\sum_{a=0}^{\infty}x_nT_n(\mathrm{X})\quad (-a<x<a), \tag{18}$$

where X is defined by (17).

Now in the expression for the pressure $p(x)$ we replace x by s and X by V, and substitute it into the right-hand side of (A8). Using the relations (B17), we get

$$\sigma_y\big|_{y=0}=\frac{l^2E}{\sqrt{2(x^2-a^2)(x^2-l^2)}}\sum_{n=0}^{\infty}(-1)^n x_n\frac{\mathrm{ch}(\pi nuK')}{\mathrm{ch}(\pi nK/K')}\quad (x>l), \tag{19}$$

where coefficients x_n are defined by formulas (12) and the variable u $(0<u<K)$ is given by the formula (B17).

Let us find the crack opening on the interval $a<x<l$. In the variables of (9), the formula (8) for dimensionless opening takes the form

$$\Psi_0(\rho\cos\vartheta)=\frac{4\rho}{\pi}\int\limits_{\alpha}^{\pi-\alpha}\ln\left[\sin\left(\frac{\vartheta+\varphi}{2}\right)\Big/\left|\sin\left(\frac{\vartheta-\varphi}{2}\right)\right|\right]\omega_0(\varphi)\,d\varphi\qquad\left(0<\vartheta<\alpha;\ \alpha=\arccos\frac{a}{l}\right).$$

Substituting in this expression $\omega_0(\vartheta)$ from (11) and taking into account the relations (B10) for $0<\vartheta<\alpha$ as well as (B7) where $\sin\varphi=sn(u,k)$, we get

$$\Psi_0(\rho\cos\vartheta)=4\rho\sum_{n=0}^{\infty}\frac{x_n}{n}\nu_n\,\mathrm{sh}\left(\frac{\pi nu}{K'}\right)\qquad(0\le\vartheta\le\alpha\quad\text{or}\quad 0\le u\le K),$$
$$u=\int\limits_0^{\sin\varphi}\frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}=F(\varphi,k);\ k=\mathrm{tg}^2\frac{\alpha}{2};\ \varphi=\arcsin\left(\frac{\mathrm{tg}(\vartheta/2)}{\mathrm{tg}(\alpha/2)}\right), \tag{20}$$

where $F(\varphi,k)$ is an incomplete elliptic integral of the first kind (B3a).

The SIF is determined by the formula (A9) (the dimensionless SIF is determined by the formula (7)) where the pressure $p(x)$ is given by (18).

Let us express characteristics of the problem through the solution of the finite system of equations (15). Then the solution of the equation (5), i.e. the dimensionless pressure of rigid inclusion on the crack edges, will be determined by the formula (16) in terms of the solution X_m $\left(m=\overline{1,N}\right)$ of the system (15). However, using the Gauss quadrature formula for integrals with the Cauchy kernel and for definite integrals, from (6)–(8) we have the following formulas for the dimensionless normal stresses, the dimensionless SIF, and dimensionless opening:

$$\left.\begin{aligned}
&\sigma_y^0(\xi)=-\sigma_0(\xi)=-\frac{1}{N\sqrt{\xi^2-\rho^2}}\sum_{m=1}^{N}\frac{\sqrt{\rho^2-\eta_m{}^2}\,\mathrm{X}_m}{\eta_m-\xi}\quad(\xi>\rho),\quad K_I^0=\frac{\pi}{N}\sum_{m=1}^{N}\sqrt{\frac{\rho+\eta_m}{\rho-\eta_m}}\mathrm{X}_m,\\
&\Psi_0(\xi)=\frac{2}{N}\sum_{m=1}^{N}\mathrm{X}_m\ln\frac{\rho^2-\xi\eta_m+\sqrt{(\rho^2-\xi^2)(\rho^2-\eta^2{}_m)}}{\rho^2-\xi\eta_m-\sqrt{(\rho^2-\xi^2)(\rho^2-\eta^2{}_m)}}\quad(1\le\xi\le\rho).
\end{aligned}\right\}\tag{21}$$

Calculate also the resultant pressure on the crack edges

$$P=\int_{-a}^{a}p(x)dx\;\Rightarrow\;P_0=\int_{-1}^{1}p_0(\xi)d\xi\;(P_0=P/aE);\qquad P_0=\frac{\pi}{N}\sum_{m=1}^{N}\mathrm{X}_m.\tag{22}$$

6. Particular cases. Let us consider two important particular cases of the discussed problem where the thin rigid inclusion has the form of a line segment, more precisely, the form of a thin rectangle of length $2a$ and height $2\delta(\delta<<a)$, and where it has the form of a strongly oblate along its major axis half-ellipse part (1).

In the case of a rectangular inclusion (Fig.2), $f(x)=\delta$ and the right-hand side of the governing integral equation (5) is $f_0(\xi)=\delta_0\;(\delta_0=\delta/a)$. Then for the right-hand side of equation (10), equivalent to (5), we also have $g_0(\vartheta)=\delta_0$. Hence by the orthogonality of the Chebyshev polynomials of the first kind (B11), it follows from (12) and (B9) that

$$x_0=\left(\delta_0\cos^4(\alpha/2)/\rho KK'\right)h_0,\quad h_0=\int_{\alpha}^{\pi-\alpha}\frac{d\vartheta}{\sqrt{\cos2\alpha-\cos2\vartheta}}\quad x_n=h_n=0\quad(n=1,2,...).$$

Calculate the integral

$$h_0=\int_{\alpha}^{\pi-\alpha}\frac{d\vartheta}{\sqrt{1+\cos2\alpha-(1+\cos2\vartheta)}}=\frac{1}{2\sqrt{2}}\int_{\alpha}^{\pi-\alpha}\left(\sin\frac{\vartheta+\alpha}{2}\sin\frac{\vartheta-\alpha}{2}\cos\frac{\vartheta+\alpha}{2}\cos\frac{\vartheta-\alpha}{2}\right)^{-1/2}d\vartheta=$$

$$=\frac{1}{\sqrt{2}\sin\alpha}\int_{\alpha}^{\pi-\alpha}\left[\left(\mathrm{ctg}^2\frac{\alpha}{2}-\mathrm{t}g^2\frac{\alpha}{2}\right)\left(\mathrm{ctg}^2\frac{\vartheta}{2}-\mathrm{tg}^2\frac{\alpha}{2}\right)\right]^{-1/2}\frac{d\vartheta}{\cos^2(\vartheta/2)}.$$

Passing to the variable y and parameters c and d by formulas (B4), we get from the above expression

$$h_0=\frac{\sqrt{2}}{\sin\alpha}\int_c^d\frac{dy}{\sqrt{(d^2-y^2)(y^2-c^2)}}.$$

Using the value of this integral [23] (p.260, f.-la 3.152.10), we finally have

$$h_0=K'/\sqrt{2}\cos^2(\alpha/2).$$

As a result,

$$x_0=\frac{(1+\rho)\delta_0}{2\sqrt{2}\rho^2K}=\frac{(a+l)\delta}{2\sqrt{2}K(k)l^2}\quad\left(k=\frac{\rho-1}{\rho+1}=\frac{l-a}{l+a}\right),\quad x_n=0\quad(n=1,2,...).\tag{23}$$

Now taking into account (23), we obtain from (18)

$$p(x)=\frac{E(a+l)\delta}{4K(k)\sqrt{(a^2-x^2)(l^2-x^2)}}\quad(-a<x<a) \tag{24}$$

or in the dimensionless form

$$p_0(\xi)=\frac{(1+\rho)\delta_0}{4K(k)\sqrt{(1-\xi^2)(\rho^2-\xi^2)}}\quad(-1<\xi<1).$$

Then with the help of (24) we calculate SIF K_I by the formula (A9):

$$K_I=\frac{1}{\pi}\sqrt{\frac{\pi}{l}}\frac{E(a+l)\delta}{4K(k)}\int_{-a}^{a}\frac{ds}{(l-s)\left(\sqrt{a^2-s^2}\right)}$$

Again, using the value of the well-known integral [24] (p. 175, f.-la (21)), we get

$$K_I=\sqrt{\frac{\pi}{l}}\frac{E\delta}{4K(k)}\sqrt{\frac{l+a}{l-a}}=\frac{aE}{\sqrt{\pi l}}K_I^0 \tag{25}$$

where K_I^0 is the dimensionless SIF:

$$K_I^0=\frac{\pi\delta_0}{4K(k)}\sqrt{\frac{\rho+1}{\rho-1}}. \tag{26}$$

By the formula (19) using (23), we find immediately the normal breaking stresses outside the crack

$$\sigma_y\Big|_{y=0}=\frac{(a+l)\delta E}{4K(k)\sqrt{(x^2-a^2)(x^2-l^2)}}\quad(x>l) \tag{27}$$

or in the dimensionless form

$$\sigma_y^0(\xi)=\frac{(1+\rho)\delta_0}{4K(k)\sqrt{(\xi^2-1)(\xi^2-\rho^2)}}\quad(\xi>\rho).$$

Now SIF K_I can be also calculated by the formula (27) that once again will lead to (25).

Finally, the dimensionless opening of crack edges outside the rigid inclusion according to (20) and (23) is determined by the formula

$$\Psi_0(\rho\cos\vartheta)=\frac{(1+\rho)\delta_0}{\rho K(k)\cos^2(\alpha/2)}u\qquad(0\le\vartheta\le\alpha),$$

$$u=\int_0^{\varphi}\frac{d\tau}{\sqrt{1-k^2\sin^2\tau}}=\int_0^{\sin\varphi}\frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}=F(\varphi,k);\ \varphi=\arcsin\left(\frac{\mathrm{tg}(\vartheta/2)}{\mathrm{tg}(\alpha/2)}\right), \tag{28}$$

where $F(\varphi,k)$ like that in (B3a) is the incomplete elliptic integral of the first kind of the modulus k.

Calculate also the resultant pressure from (24):

$$P=\int_{-a}^{a}p(x)\,dx=aE\int_{-1}^{1}p_0(\xi)\,d\xi=\frac{aE(1+\rho)\delta_0}{2K(k)}\int_{0}^{1}\frac{d\xi}{\sqrt{(1-\xi^2)(\rho^2-\xi^2)}}.$$

The last integral is the complete elliptic integral of the first kind of module $\chi=a/l$. Hence,

$$P=\frac{E(a+l)\delta}{2lK(k)}K(\chi)\qquad\left(k=\frac{\rho-1}{\rho+1}=\frac{1-\chi}{1+\chi}\right).\tag{29}$$

Relationship (29) establishes the dependence between the resultant pressure P and the half-width of inclusion δ, i.e. the measure of the vertical settlement of longitudinal sides of a rectangular inclusion into an elastic matrix. In the dimensionless form, we get

$$P_0=(1+\chi)\delta_0K(\chi)/2K(k)\ (P_0=P/aE).\tag{30}$$

Consider the second particular case. Equation of the upper oblate semi-ellipse (1) is represented in the form

$$f_0(\xi)=f(a\xi)/a=\varepsilon\sqrt{a_0^2-\xi^2}\quad(\varepsilon=b_1/a_1;\ a_0=a_1/a;\ \varepsilon<<1).\tag{31}$$

Then all mechanical characteristics of the problem may be calculated by formulas (16) and (21) - (22).

Let us discuss the limiting case $a\to l$, assuming that a thin absolutely rigid elliptical inclusion, indenting into a crack across its entire surface, is closely adjacent to the edges of the crack along its entire length. In this case, in (31) $\varepsilon=b_1/l$, $a_0=1$ should be taken and based on the integral relationship (B19), the solution to the governing integral equation (5), where $\rho=1$, should be represented in the form of an infinite series (13) of the Chebyshev polynomials of the second kind with unknown coefficients y_n. These coefficients are expressed by (14), from which for the function (31) we have

$$y_1=\varepsilon/2,\quad y_n=0\quad(n=1,2,...).$$

Hence by (13) $p_0(\xi)=\varepsilon/2\quad(-1\le\xi\le1)$ or

$$p(x)=\varepsilon E/2\quad(-l\le x\le l).\tag{32}$$

Substituting (32) into (A8) and taking into account the value of the well-known integral [24] (p.175, f.-la (19)), we easily obtain

$$\sigma_y\big|_{y=0}=\frac{\varepsilon E}{2\sqrt{x^2-l^2}}\left(x-\sqrt{x^2-l^2}\right)\ (x>l).\tag{33}$$

Calculate the SIF K_I from (33):

$$K_I=\lim_{x\to l+0}\left[\sqrt{2\pi(x-l)}\,\sigma_y\big|_{y=0}\right]=\varepsilon E\sqrt{\pi l}/2\tag{34}$$

or in the dimensionless form

$$K_I^0=\varepsilon/2;\quad K_I^0=K_I/E\sqrt{\pi l}.$$

The same result (34) can be obtained directly by means of (32) and (A9) for $a=l$.

For a comparative analysis of the analytical expression of SIF (25) and the other known similar expressions, we consider the following cases.

1) As a comparison, we consider the case of a normal opening of a crack by a dipole of concentrated at the origin forces with a magnitude P determined by (29). In this case, for SIF we have [20]

$$\tilde{K}_I = P/\sqrt{\pi l}.$$

Then writing the SIF (25) with the help of (29) as

$$K_I = \pi P/2\sqrt{\pi l}K(\chi)\sqrt{1-\chi^2},$$

we get for their ratio

$$k_1 = k_1(\chi) = K_I/\tilde{K}_I = \pi/2\sqrt{1-\chi^2}K(\chi). \quad (35)$$

2) With the same case of dipole forces we compare SIF (7) when the inclusion shaped as a strongly oblate half-ellipse (31). In this case, taking into account (22), we have

$$k_2 = k_2(\chi) = \frac{K_I}{\tilde{K}_I} = \sum_{m=1}^{N}\sqrt{\frac{1+\chi\eta_m}{1-\chi\eta_m}}\mathrm{X}_m\left(\sum_{m=1}^{N}\mathrm{X}_m\right)^{-1}, \quad (36)$$

where X_m is the solution of the system (15).

3) We also compare SIF (25) with SIF calculated by the model proposed in [7]. We assume that normal forces $p(x)$ are symmetrically applied to the crack edges along the segment $(-a,a)$. In accordance with Sadovsky's solution to the problem on the indentation of a punch with a flat base into the elastic half-plane [25] we have:

$$p(x) = P/\pi\sqrt{a^2-x^2} \quad (-a<x<a).$$

Then by (7)

$$\tilde{K}_I = \frac{1}{\sqrt{\pi l}}\frac{P}{\pi}\int_{-a}^{a}\sqrt{\frac{l+s}{l-s}}\frac{ds}{\sqrt{a^2-s^2}} = \frac{2P}{\pi\sqrt{\pi l}}K(\chi)$$

and hence

$$k_3 = k_3(\chi) = \frac{K_I}{\tilde{K}_I} = \pi^2/4K^2(\chi)\sqrt{1-\chi^2}. \quad (37)$$

7. On estimation of crack resistance of a plate with a crack. Now proceeding from (25), we note that $\lim_{a\to l}K_I = \infty$, i.e. when the inclusion ends are approaching the crack tips, the SIF K_I increases infinitely, taking on also its critical value K_{IC} at which the crack starts to propagate. It follows that the crack will propagate before inclusion ends reach the tips of the crack. This phenomenon is quite similar to the phenomenon of cracking the brittle elastic bodies during their wedging by an absolutely rigid thin wedge [12]. The quantity K_{IC} is called the crack resistance or the limit of the ductile fracture of materials during normal separation at the maximum constraint of plastic deformation and is an important characteristic of materials. The values of K_{IC} for a large number of materials are given in [26].

With the help of (25), we evaluate the critical value of the parameter $\chi = \rho^{-1} = a/l$, at which the crack begins to propagate. For this, we require the fulfillment of the condition

$$K_I \geq K_{Ic} \Rightarrow \sqrt{\frac{\pi}{l}}\frac{E\delta}{4K(k)}\sqrt{\frac{l+a}{l-a}} \geq K_{Ic}.$$

From this

$$\sqrt{\frac{l+a}{l-a}} \geq K_{IC}\frac{\sqrt{l}}{\sqrt{\pi}}\frac{4K(k)}{E\delta}. \tag{38}$$

Evaluate the complete elliptic integral of the first kind $K(k)$. It is evident that

$$k' = \sqrt{1-k^2} \leq \sqrt{1-k^2t^2} \leq 1 \quad (0 \leq t \leq 1) \quad \Rightarrow \qquad 1 \leq \frac{1}{\sqrt{1-k^2t^2}} \leq \frac{1}{k'}.$$

Hence

$$\frac{\pi}{2} \leq K(k) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} \leq \frac{\pi}{2k'}.$$

If we now require the fulfillment of the condition

$$\sqrt{\frac{l+a}{l-a}} \geq \frac{4K_{IC}\sqrt{l}}{\sqrt{\pi}E\delta}\frac{\pi}{2k'},$$

then the condition (38) is a fortiori fulfilled. Further elementary transformation of this inequality leads to a quadratic inequality for a:

$$K_0^2a^2 + a - l^2K_0^2 \geq 0, \quad K_0 = \sqrt{\pi}K_{IC}/E\delta.$$

The solution to this inequality has the forms

$$a \geq \frac{-1+\sqrt{1+4l^2K_0^4}}{2K_0^2} \quad \Rightarrow \quad \frac{a}{l} \geq \frac{-1+\sqrt{1+4l^2K_0^4}}{2K_0^2 l}.$$

Finally, we have

$$\chi = a/l \geq \chi_c,\ \chi_c = \left[-1+\sqrt{1+\left(K_{IC}^0\right)^2}\right] \Big/ K_{IC}^0 = = K_{IC}^0 \Big/ \left[1+\sqrt{1+\left(K_{IC}^0\right)^2}\right]; K_{IC}^0 = 2\pi l K_{IC}^2 / E^2\delta^2. \tag{39}$$

We will call the dimensionless quantity K_{Ic}^0 the reduced crack resistance.

At $\chi \geq \chi_c$ the crack propagation begins and brittle fracture of the plate occurs. Hence, it follows that according to condition (39), a plate with a crack of a given length can withstand a thin rectilinear inclusion indented into crack edges only if the inclusion has proper length. Consequently, a complete contact of the rectilinear thin inclusions with the crack edges along its entire length from the point of view of fracture mechanics is impossible. Therefore, it is necessary to introduce corresponding corrections in the formulation of the Sherman-Muskhelishvili classical mixed boundary-value problem [19] (pp. 444-446), where full contact is considered. This issue was studied in detail in [6].

8. Numerical analysis of the mechanical characteristics. To determine changes in the basic mechanical quantities of the discussed problem and to reveal regularities of their changes depending on specific geometrical and physical parameters, the numerical implementation of the obtained analytical results were carried out.

In the case of a thin rectilinear inclusion (Fig. 2), values of the dimensionless SIF K_I^0 are calculated by the exact formula (26) at $\delta_0 = 0,05$, and for different values of the parameter χ (29), the relative distance from the inclusion right end to the crack right tip $(\chi = a/l)$. At the same time for the same values of parameters, K_I^0 has been also calculated by the high-accuracy approximate formula (21), where X_m is the solution to the linear system of equations (15). The calculation results obtained by both formulas, which practically coincide for large N, are represented in Table 1.

Table 1. Values of K_I^0 (exact and approximate)

χ	0,03	0,05	0,09	0,1	0,2	0,5	0,8	0,9
K_I^0	0,0161 0,0161	0,0179 0,0179	0,0207 0,0207	0,0214 0,0214	0,0266 0,0266	0,0421 0,0421	0,0748 0,0751	0,1089 0,1096

The Table of exact and approximate values of the dimensionless SIF K_I^0 for different values of the parameter $\chi = a/l$ and for a fixed value of $\delta_0 = \delta/a = 0,05$ in case of the thin linear smooth rigid inclusion, wherein the upper row shows the exact values of K_I^0, and the bottom row – approximate values of K_I^0.

Here, in the first row values calculated by the formula (26), in the second row those calculated by (21) are given. According to these values, for the sake of visual illustration of the change of dimensionless SIF, the graph of K_I^0 is plotted (Fig.3). It shows that the value of K_I^0 increases significantly as $\chi \to 1$.

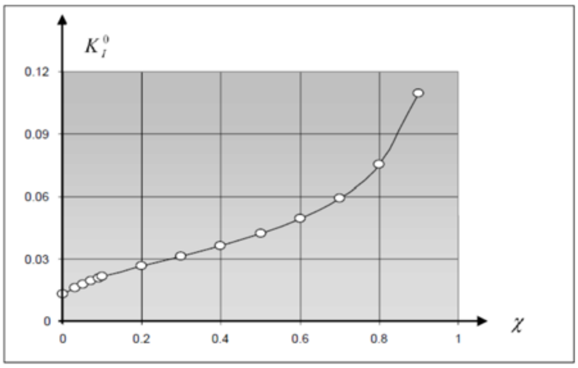

Fig.3. The graph of dimensionless SIF, K_I^0 depending on the parameter χ $(\chi = a/l)$ and for a fixed value of δ_0 $(\delta_0 = \delta/a = 0,05)$.

Turn to the quantity χ_c defined by the formula (39) and consider a specific calculation example to determine the order of magnitude of K_I^0. Let the elastic plate be made of extruded aluminum strip alloys for which according to [26] (p.113, Table 2.2) $K_{Ic} = 410 MPa \cdot \sqrt{cm}$ and according to [27] (p.63, Table 1) $E = 0,7 \cdot 10^6 \text{kg/cm}^2 = = 6,9 \cdot 10^4 MPa$. Assume that $2l = 40cm$ and $\delta = 0,1cm$; 0,3cm; 0,5cm; 0,7cm; 1cm; 2cm; 5cm; 10cm. For these values of physical constants and geometrical parameters, the critical value of χ at which the crack propagates, χ_c, as well as the reduced crack resistance K_{Ic}^0 are calculated from the formulas (39). According to the results of calculations, Table 2 was compiled. As it follows from Table 2, the values of K_{Ic}^0 are small numbers (e.g. for $\delta = 1cm$ $K_{Ic}^0 = 4,435 \cdot 10^{-3}$), and therefore, we can simplify (39) by taking $1+\left(K_{Ic}^0\right)^2 \approx 1$. As a result, we can practically assume that $\chi_c \approx K_{Ic}^0/2$ which is confirmed by the first row of Table 2.

Table 2. Values of χ_c and K_{Ic}^0

δ/l	0,005	0,015	0,035	0,05	0,1	0,25	0,5
χ_c	0.211885	0.02463	0.00453	0.00222	0.00055	0.00008	0.00002
K_{Ic}^0	0.44369	0.04929	0.00905	0.00443	0.00111	0.00018	0.00004

Table of values of the parameter χ_c, the critical value of the relative distance from the right end of linear inclusion to the right crack tip ($\chi = a/l$) and the reduced crack resistance K_{Ic}^0 for different values of δ/l .

Note that the results of calculations of the dimensionless resultant P_0 according to the formulas (22) and (30) also coincide with high accuracy, and the value of P_0 increases appreciably with increasing of χ .

Next, the numerical realization of (35)-(37) is carried out and results of calculating the deviations of the compared SIFs k_j and their errors $\left|1-k_j\right|$ $(j=1,2,3)$ depending on the values of the characteristic parameter are given in Tables 3 and 4 ($a_0 = 1.2$ and $a_0 = 2$ are taken when calculating k_2)

Table 3. Values of k_j and errors $\left|1-k_j\right|$ $(j=1,2,3;\ a_0=1.2)$

χ	k_1	$\left\|1-k_1\right\|$	k_2	$\left\|1-k_2\right\|$	k_3	$\left\|1-k_3\right\|$
0.01	0.997542	0.00245798	0.999983	0.0000165878	0.99504	0.00495967
0.1	0.97908	0.0209196	0.999947	0.0000532126	0.953793	0.0462066
0.2	0.965995	0.0340055	1.00176	0.00176355	0.914292	0.085708
0.3	0.960763	0.0392368	1.00682	0.00681802	0.880549	0.119451
0.4	0.964197	0.0358027	1.01642	0.0164202	0.852062	0.147938
0.5	0.978277	0.0217226	1.03251	0.0325148	0.828809	0.171191
0.6	1.00714	0.00714397	1.05857	0.0585711	0.811471	0.188529
0.7	1.05984	0.0598409	1.10182	0.101817	0.80217	0.19783
0.8	1.15984	0.159839	1.18098	0.180976	0.807135	0.192865
0.9	1.3978	0.397799	1.3702	0.370199	0.85166	0.14834

Table of the values of compared SIFs k_j and their absolute errors $\left|1-k_j\right|$ $(j=1,2,3)$ for different values of the parameter χ , where $a_0 = 1.2$.

As follows from (35) - (37), $k_j(0)=1$ and, therefore, the quantities $\left|1-k_j\right| \; (j=1,2,3)$ are the absolute deviation k_j from the unit. These values simultaneously give the relative errors of the compared

SIFs, since according to the cases discussed above $\left|1-k_j\right| = \left|K_I - \tilde{K}_I\right| / \tilde{K}_I$.

Analysis of the data in Table 3 show that for small values of χ, the errors $\left|1-k_j\right| \; (j=1,2,3)$ are very small numbers. For instance, for $\chi = 0,5$ these errors are about 3%, for $\chi < 0,5$ – even less. The highest accuracy is provided in cases 1) and 2) (section 6). The values of $\left|1-k_j\right| \; (j=1,2,3)$ increase significantly with the increase of χ and a_0.

We also present the results of the numerical analysis that illustrate the course of the change in the crack opening, which can be used in deformation theories of cracks propagation. Values of the dimensionless crack opening on the interval $a \le x \le l$ or on the corresponding intervals $\rho \le \xi \le 1$ and $0 \le \vartheta \le \alpha$ $(\alpha = \arccos \alpha = \arccos \alpha / l)$ are calculated by the exact formulas (28) and by the approximate formula (21) $(\xi = \rho \cos \vartheta)$ at nodal points $\vartheta_j = j\alpha/n \; \left(j=\overline{0,n}\right)$ of the interval $0 \le \vartheta \le \alpha$. The graph of the change of $\Psi_0(\xi)$ depending on ξ for $\delta_0 = 0.05$ and $\chi = 0.5$ is shown in Fig. 4, in which points corresponding to exact and approximate solutions practically merged. $\Psi_0(\xi)$ reaches its highest value at the point $\xi = 1 \; (x=a)$, i.e. at the right end of the inclusion.

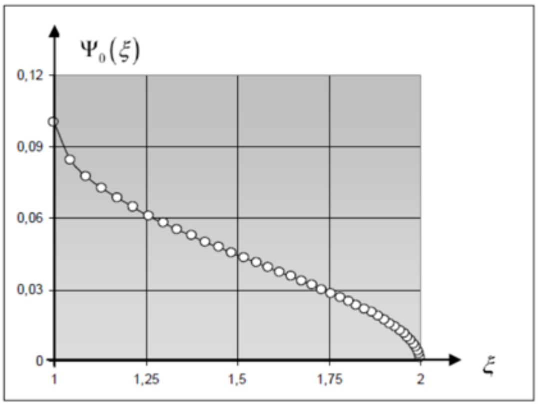

Fig.4. Graph of the dimensionless crack opening $\Psi_0(\xi)$ on the interval $1 \le \xi \le \rho = l/a \; (a \le x \le l)$ in case of a linear inclusion for fixed values $\delta_0 = \delta/a = 0,05$ and $\chi = a/l = 0,5$.

We turn now to the second particular case when a thin inclusion has the shape of a strongly oblate along its length ellipse (31). In this case, by solving a linear system of algebraic equations (15) the main mechanical characteristics are expressed by formulas (21)–(22). For different values of the parameter χ and for $\varepsilon = 0.05$, $a_0 = 1.4$, the values of dimensionless SIF K_I^0 are calculated by the formula (21) and using these values, the graph of SIF change is plotted in Fig. 5.

Note that values of K_I^0 increase appreciably with the increase of the parameter χ . It is also confirmed analytically. Indeed, it follows from formulas (C1) - (C2) of Appendix C that $\lim\limits_{\rho\to 1} K_I^0 = \lim\limits_{\chi\to 1}\left[BK(\chi)\right] = \infty$, i.e. the values of K_I^0 as $\chi \to 1$ not only increase but increase infinitely. Now, it should be emphasized that at the end of section 6 the passage to the limit was formally carried out, $\chi \to 1(a \to l)$, i.e. it was formally assumed that the rigid thin elliptical inclusion of length $2l$ entirely fits into the crack along its entire length, which is also equal to $2l$. This problem as a mixed boundary value problem of the mathematical theory of elasticity is correct and has a simple solution with characteristics (32)-(34). However, from the point of view of fracture mechanics, this solution is unfounded and devoid of real physical content since $K_I^0 \to \infty$ as $\rho \to 1$ and therefore when $\rho \to 1$ the crack propagates earlier than the end points of the thin inclusion reach the tips of the crack.

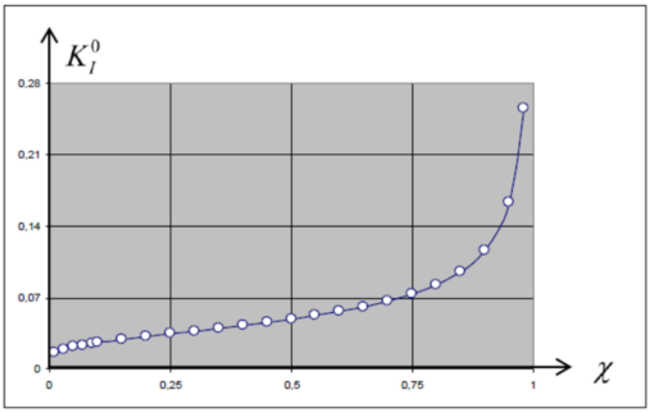

Fig.5. The graph of the change of dimensionless SIF, K_I^0 , depending on the parameter χ for $\varepsilon = 0.05,\ \ a_0 = 1.4$ in case of a rigid strongly oblate elliptical inclusion

8. Conclusions. In the paper, by the method of integral equations an exact solution to the plane problem of the theory of elasticity on the stress state of an infinite elastic plate with a finite rectilinear crack, in which a thin smooth absolutely rigid inclusion shaped as an oblate along its length ellipse is indented, is obtained.

At the same time, an approximate solution of the problem is obtained by reduction of the governing integral equation to a system of linear algebraic equations. The main mechanical quantities and characteristics of fracture mechanics are represented by explicit analytical formulas of simple structures.

The important special cases of rigid inclusions shaped as a thin rectangle and as a strongly oblate ellipse are considered; to reveal and promote understanding the regularities of the change in the characteristics of fracture mechanics, their numerical analysis is carried out.

It is established that the crack extension occurs when the inclusion end points are approaching the crack tips, while the crack propagates earlier than the end points of inclusion will reach the crack tips. Proceeding from this phenomenon, an estimate for the crack resistance of a plate with a crack is given; namely, the critical value of the relative length of the inclusion at which the crack starts to spread is determined.

It is shown that the model of the complete contact of the inclusion along the entire crack length, adopted in the classical boundary-value problems of the theory of elasticity, is unacceptable from the point of view of fracture mechanics.

Appendix A. Let us consider an auxiliary boundary-value problem of the stress state of an infinite elastic plate with a crack along the line segment $-l \le x \le l$, to the upper and lower edges of which normal distributed forces of intensity $p(x)$, equal in magnitude but opposite in direction, are applied and $p(-x) = p(x)$, $p(x) \equiv 0$ for $a < |x| < l$.

Due to the symmetry about the abscissa axis, this auxiliary problem is equivalent to the following mixed boundary-value problem for the upper elastic half-plane:

$$\left.\begin{aligned} &\sigma_y\big|_{y=+0} = -p(x)\ \ (-l<x<l);\ \ \tau_{xy}\big|_{y=+0} = 0\ \ (-\infty<x<\infty);\ \ \mathrm{v}\big|_{y=+0} = 0\ \ (|x| \ge l);\\ &\sigma_x,\ \sigma_y,\ \tau_{xy} \to 0\ \ as\ \ x^2+y^2 \to \infty \end{aligned}\right\} \tag{A1}$$

A solution of the problem (A1) in displacement can be immediately obtained using the solution of the well-known Flamant's problem for an elastic half-plane in combination with the linear superposition principle. Namely, introducing the notation

$$\sigma_y\big|_{y=+0} = -\sum(x) = \begin{cases} -p(x) & (-l<x<l),\\ -\sigma(x) & (|x|>l) \end{cases} \tag{A2}$$

and using the known formula [25, pp.95-96] for vertical displacements of boundary points of the upper elastic half-plane (upper semi-infinite plate) we have

$$\mathrm{v}(x,+0) = \frac{2}{\pi E}\int_{-\infty}^{\infty} \ln\frac{1}{|x-s|}\sum(s)\,ds + \mathrm{const} \qquad (-\infty<x<\infty).$$

By differentiation of both sides of the equation with respect to x we get

$$\frac{E}{2}\psi(x) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sum(s)\,ds}{s-x},\quad \psi(x) = \frac{d\mathrm{v}(x,+0)}{dx} \qquad (-\infty<x<\infty). \tag{A3}$$

From (A3) by Hilbert's inversion formula, we will come to the key equation of the problem taking into account the boundary condition (2b):

$$\sum(x) = -\frac{E}{2\pi}\int_{-\infty}^{\infty}\frac{\psi(s)\,ds}{s-x} = -\frac{E}{2\pi}\int_{-l}^{l}\frac{\psi(s)\,ds}{s-x} \quad (-\infty<x<\infty). \tag{A4}$$

Now considering the key equation (A4) on the interval $(-l,l)$, in accordance with (A2) we can write

$$p(x) = -\frac{E}{2\pi}\int_{-l}^{l}\frac{\psi(s)\,ds}{s-x} \quad (-l<x<l). \tag{A5}$$

Then the equation (A5) is treated as singular integral equation (SIE) and the following well-known formula from [28] (pp.445-446) is used:

$$\psi(x) = \frac{d\mathrm{v}(x,+0)}{dx} = \frac{2}{\pi E}\frac{1}{\sqrt{l^2-x^2}}\int_{-a}^{a}\frac{\sqrt{l^2-s^2}\,p(s)\,ds}{s-x} + \frac{C}{\sqrt{l^2-x^2}}. \tag{A6}$$

Integrating both sides of this equation and using the well-known integral expression given in [29] (p.111), we obtain

$$\mathrm{v}(x,+0)=\frac{1}{\pi E}\int_{-a}^{a}\ln\frac{l^2-xs+\sqrt{(l^2-x^2)(l^2-s^2)}}{l^2-xs-\sqrt{(l^2-x^2)(l^2-s^2)}}p(s)\,ds+$$

$$+C\arcsin\frac{x}{a}+C_1\quad(-l\le x\le l),$$

where C and C_1 are constants. Since $\mathrm{v}(\pm l,+0)=0$, then $C=C_1=0$.

Consequently,

$$\mathrm{v}(x,+0)=\frac{1}{\pi E}\int_{-a}^{a}\ln\frac{l^2-xs+\sqrt{(l^2-x^2)(l^2-s^2)}}{l^2-xs-\sqrt{(l^2-x^2)(l^2-s^2)}}p(s)\,ds\quad(-l\le x\le l).\tag{A7}$$

Formula (A7) coincides with the result of the monograph [30, p.33] obtained earlier by the same author with the help of complex potentials of the plane theory of elasticity.

Considering the key equation (A4) outside the interval $(-l,l)$, we get an expression of normal stresses outside the crack along its location line:

$$\sigma_y\big|_{y=+0}=-\sigma(x)=\frac{E}{2\pi}\int_{-l}^{l}\frac{\psi(s)\,ds}{s-x}\quad(|x|>l).$$

Substituting the expression $\psi(x)$ from (A6) where $C=0$, after simple transformations we obtain

$$\sigma_y\big|_{y=+0}=-\sigma(x)=-\frac{\mathrm{sign}x}{\pi\sqrt{x^2-l^2}}\int_{-a}^{a}\frac{\sqrt{l^2-s^2}\,p(s)\,ds}{s-x}\quad(|x|>l).\tag{A8}$$

Here, the expression for the known integral [24] (p.175) was used.

Proceeding from (A8), we calculate the SIF K_I and due to the symmetry we restrict our consideration only by its value at the right tip of the crack, $x=l$:

$$K_I=\lim_{x\to l+0}\left[\sqrt{2\pi(x-l)}\sigma_y(x,+0)\right]=\frac{1}{\sqrt{\pi l}}\int_{-a}^{a}\sqrt{\frac{l+s}{l-s}}p(s)\,ds.\tag{A9}$$

Formula (A9) coincides with the known result obtained in [10].

Finally, again taking into account the symmetry, we get from (A7) the following expression for the crack opening out of inclusion, calculated only for the right segment $a\le x\le l$:

$$\Psi(x)=2\mathrm{v}(x,+0)=\frac{2}{\pi E}\int_{-a}^{a}\ln\frac{l^2-xs+\sqrt{(l^2-x^2)(l^2-s^2)}}{l^2-xs-\sqrt{(l^2-x^2)(l^2-s^2)}}p(s)\,ds\quad(a\le x\le l).\tag{A10}$$

Appendix B. To construct the exact solution of Eq. (10), let us find eigenfunctions and eigenvalues of the integral operator

$$Kh(\vartheta)=\frac{1}{\pi}\int_{\alpha}^{\beta}\ln\left(\sin\left(\frac{\vartheta+\varphi}{2}\right)\Big/\left|\sin\left(\frac{\vartheta-\varphi}{2}\right)\right|\right)h(\varphi)\,d\varphi.$$

For this purpose, we use the results obtained in [31] and in [32], where by the methods of logarithmic potential and with the help of conformal mapping of the complex plane with two identical and symmetrically located cuts onto an annular ring, the spectral relationships are established by means of Jacobi elliptic sine functions:

$$\left.\begin{array}{l}\dfrac{1}{\pi}\displaystyle\int_{c}^{d}\ln\frac{y+\mathrm{v}}{|y-\mathrm{v}|}\frac{T_n(\mathrm{V})\,d\mathrm{v}}{\sqrt{(d^2-\mathrm{v}^2)(\mathrm{v}^2-c^2)}}=\lambda_n T_n(Y)\quad(c<y<d;\ 0<c<d;\ n=0,1,2,...)\\ \mathrm{V}=\cos\Phi,\quad \Phi=\dfrac{\pi}{K'}\displaystyle\int_{1}^{\mathrm{v}/c}\frac{dt}{\sqrt{(t^2-1)(1-k^2t^2)}};\quad k=\dfrac{c}{d};K=K(k)=\displaystyle\int_{0}^{1}\frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}\\ Y=\cos\Theta,\quad \Theta=\dfrac{\pi}{K'}\displaystyle\int_{1}^{y/c}\frac{dt}{\sqrt{(t^2-1)(1-k^2t^2)}};\lambda_n=\begin{cases}\dfrac{1}{\pi n\,d}K'\,\mathrm{th}(\pi nK/K')(n=1,2,...);\\ K/d\qquad\qquad(n=0).\end{cases}\end{array}\right\}\quad\text{(B1)}$$

Here $K(k)$ is the complete elliptic integral of the first kind of the modulus k, $K'=K(k')$, where $k'=\sqrt{1-k^2}$ is the complementary modulus; $T_n(\mathrm{X})$ are Chebyshev polynomials of the first kind of the argument X; Φ and Θ are incomplete elliptic integrals of the first kind. In addition, the following integral relations cognate with (B1) were also obtained in [31,32]:

$$\frac{1}{\pi}\int_{c}^{d}\ln\frac{y+\mathrm{v}}{|y-\mathrm{v}|}\frac{T_n(\mathrm{V})\,d\mathrm{v}}{\sqrt{(d^2-\mathrm{v}^2)(\mathrm{v}^2-c^2)}}=$$

$$=\frac{K'}{\pi dn\,\mathrm{ch}(\pi nK/K')}\Big[H(c-y)+(-1)^n H(y-d)\Big]\mathrm{sh}(\pi nu/K')\qquad\text{(B2)}$$

$$\big(y\in(0,c)\cup(d,\infty);\ n=0,1,2,...\ 0<u<K\big)$$

where $H(x)$ is the well-known Heaviside function. If in (B2) $y\in(0,c)$, then $y=c\,\mathrm{sn}(u,k)$ and if $y\in(d,\infty)$, then $y=d/\mathrm{sn}(u,k)$ where $\mathrm{sn}(u,k)$ is the Jacobi elliptic sine function of the modulus k.

Note that in the theory of elliptic functions [23] the quantity u, called the argument $(u=\arg\varphi)$, is given by the formula

$$u=\int_{0}^{\varphi}\frac{d\tau}{\sqrt{1-k^2\sin^2\tau}}\qquad(0<k<1)$$

where the limit φ, called the amplitude $(\varphi=am\ u)$, is considered as a function of u.

Supposing $t=\sin\tau$, we get

$$u=\int_0^{\sin\varphi}\frac{dt}{\sqrt{\left(1-t^2\right)\left(1-k^2t^2\right)}}=F\left(\varphi,k\right),\tag{B3a}$$

where $F\left(\varphi,k\right)$ is the incomplete elliptic integral of the first kind. However, by definition $\sin\varphi=\mathrm{sn}\left(u,k\right)=snu$. As a result, from (B3a) the well-known formula is obtained [23] (p.924, f.-la 8.141):

$$u=\int_0^{snu}\frac{dt}{\sqrt{\left(1-t^2\right)\left(1-k^2t^2\right)}}.\tag{B3b}$$

Now, in the relations (B1) we pass to new variables ϑ,φ and parameters α,β

$$y=\mathrm{tg}\frac{\vartheta}{2},\quad \mathrm{v}=\mathrm{tg}\frac{\varphi}{2};\quad c=\mathrm{tg}\frac{\alpha}{2},\quad d=\mathrm{tg}\frac{\beta}{2}\ \left(\alpha<\vartheta,\quad \varphi<\beta\right).\tag{B4}$$

By corresponding replacements in (B1) and slightly changing notations, after simple transformations we come to the following spectral relationships:

$$\left.\begin{aligned}
&\frac{1}{\pi}\int_\alpha^\beta \ln\left[\frac{\sin\left(\vartheta+\varphi\right)}{2}\bigg/\left|\frac{\sin\left(\vartheta-\varphi\right)}{2}\right|\right]\frac{T_n\left(Y\right)d\varphi}{\sqrt{\left(\cos\varphi-\cos\beta\right)\left(\cos\alpha-\cos\varphi\right)}}=\\
&=\lambda_n\sec\frac{\alpha}{2}\sec\frac{\beta}{2}T_n\left(\mathrm{X}\right)\\
&\mathrm{X}=\cos\Theta,\quad \Theta=\frac{\pi}{K'}\cos\frac{\alpha}{2}\sin\frac{\beta}{2}\int_\alpha^\vartheta\frac{dt}{\sqrt{\left(\cos\alpha-\cos t\right)\left(\cos t-\cos\beta\right)}},\\
&Y=\cos\Phi,\quad \Phi=\frac{\pi}{K'}\cos\frac{\alpha}{2}\sin\frac{\beta}{2}\int_\alpha^\varphi\frac{dt}{\sqrt{\left(\cos\alpha-\cos t\right)\left(\cos t-\cos\beta\right)}}\\
&\left(n=0,1,2,...\right)\qquad \left(\alpha<\vartheta,\ \varphi<\beta,\ k=\mathrm{tg}\frac{\alpha}{2}\mathrm{ctg}\frac{\beta}{2}\right).
\end{aligned}\right\}\tag{B5}$$

Passing to the variables (B4) in the integral relations (B2), we get

$$\begin{aligned}
&\frac{1}{\pi}\int_\alpha^\beta\ln\left(\sin\left(\frac{\vartheta+\varphi}{2}\right)\bigg/\left|\sin\left(\frac{\vartheta-\varphi}{2}\right)\right|\right)\frac{T_n\left(Y\right)d\varphi}{\sqrt{\left(\cos\alpha-\cos\varphi\right)\left(\cos\varphi-\cos\beta\right)}}=\\
&=\frac{\sec\left(\alpha/2\right)\mathrm{cosec}\left(\beta/2\right)K'}{\pi n\,\mathrm{ch}\left(\pi nK/K'\right)}\left[H\left(\alpha-\vartheta\right)+\left(-1\right)^nH\left(\vartheta-\beta\right)\right]\mathrm{sh}\left(\pi nu/K'\right)\\
&\left(\vartheta\in\left(0,\alpha\right)\cup\left(\beta,\pi\right);\quad n=0,1,2...\quad 0<u<K\right)
\end{aligned}\tag{B6}$$

Here according to the above

$$\vartheta=\begin{cases}2\,\mathrm{arctg}\left(\mathrm{tg}\dfrac{\alpha}{2}\mathrm{sn}(u,k)\right) & (0<\vartheta<\alpha);\\ 2\,\mathrm{arctg}\left(\mathrm{tg}\dfrac{\beta}{2}\Big/\mathrm{sn}(u,k)\right) & (\beta<\vartheta<\pi).\end{cases}\tag{B7}$$

Then we write the orthogonality conditions of Chebyshev polynomials of the first kind entered in the relations (B5)

$$\int_c^d T_n(Y)T_m(Y)\frac{dy}{\sqrt{(d^2-y^2)(y^2-c^2)}}==\frac{K'}{\pi d}\int_0^\pi \cos n\Phi\cos m\Phi d\Phi=\frac{K'}{\pi d}\begin{cases}\pi & (m=n=0);\\ \dfrac{\pi}{2} & (m=n\neq 0);\\ 0 & (m\neq n).\end{cases}$$

$$(m,n=0,1,2,...).$$

Here we used the integral value from [23], (p.260, f-la 3.152.10).

Hence, the orthogonality conditions are given in the form

$$\int_c^d T_n(Y)T_m(Y)\frac{dy}{\sqrt{(d^2-y^2)(y^2-c^2)}}=\begin{cases}K'/d & (m=n=0),\\ K'/2d & (m=n\neq 0),\\ 0 & (m\neq n).\end{cases}$$

Transforming these conditions into variables (B4), we obtain

$$\int_\alpha^\beta T_n(\mathrm{X})T_m(\mathrm{X})\frac{d\vartheta}{\sqrt{(\cos\alpha-\cos\vartheta)(\cos\vartheta-\cos\beta)}}=\sec\frac{\alpha}{2}\sec\frac{\beta}{2}\begin{cases}K'/d & (m=n=0);\\ K'/2d & (m=n\neq 0);\\ 0 & (m\neq n).\end{cases}\tag{B8}$$

Note that the relations (B5)–(B7) take place on a more general interval (α,β) $(0<\alpha<\beta<\pi)$ (α,β) $(0<\alpha<\beta<\pi)$ than the interval in the integral governing IE (10) where $\beta=\pi-\alpha$. Therefore, these relations should be modified in the equation (10) assuming $\beta=\pi-\alpha$. Then after simple transformations, the spectral relations (B5) turn to the followings:

$$\left.\begin{array}{l}\dfrac{1}{\pi}\displaystyle\int_\alpha^{\pi-\alpha}\ln\left(\sin\left(\frac{\vartheta+\varphi}{2}\right)\Big/\left|\sin\left(\frac{\vartheta-\varphi}{2}\right)\right|\right)\frac{T_n(Y)d\varphi}{\sqrt{\cos 2\alpha-\cos 2\varphi}}=\mu_n T_n(\mathrm{X});(\alpha<\vartheta<\pi-\alpha;n=0,1,2...)\\ \mathrm{X}=\cos\Theta,\ \Theta=\dfrac{\pi\sqrt{2}}{K'}\cos^2\dfrac{\alpha}{2}\displaystyle\int_\alpha^\vartheta\frac{dt}{\sqrt{\cos 2\alpha-\cos 2t}};\ Y=\cos\Phi,\ \Phi=\dfrac{\pi\sqrt{2}}{K'}\cos^2\dfrac{\alpha}{2}\int_\alpha^\varphi\frac{dt}{\sqrt{\cos 2\alpha-\cos 2t}}\\ \mu_n=\begin{cases}\dfrac{K'\sec^2\frac{\alpha}{2}}{\pi\sqrt{2}\,n}\mathrm{th}\left(\dfrac{\pi nK}{K'}\right) & (n=1,2,...);\\ \dfrac{K\sec^2\frac{\alpha}{2}}{\sqrt{2}\,n}\mathrm{th}\left(\dfrac{\pi nK}{K'}\right) & (n=0);\end{cases}\quad\left(k=\mathrm{tg}^2\dfrac{\alpha}{2};\ \alpha<\vartheta,\ \ \varphi<\pi-\alpha\right)\end{array}\right\}\tag{B9}$$

and integral relations (B6) turn to:

$$\frac{1}{\pi}\int_{\alpha}^{\pi-\alpha}\ln\left[\sin\left(\frac{\vartheta+\varphi}{2}\right)\Big/\left|\sin\left(\frac{\vartheta-\varphi}{2}\right)\right|\right]\frac{T_n(Y)\,d\varphi}{\sqrt{\cos 2\alpha-\cos 2\varphi}}=$$

$$=\frac{v_n}{n}\left[H(\alpha-\vartheta)+(-1)^n H(\vartheta-\pi+\alpha)\right]\operatorname{sh}\left(\frac{\pi nu}{K'}\right)\ \left(\vartheta\in(0,\alpha)\cup(\pi-\alpha,\pi);=0,1,2...\right),$$

$$v_n=K'\sec^2\frac{\alpha}{2}\Big/\pi\sqrt{2}\operatorname{ch}\left(\pi nK/K'\right), \tag{B10}$$

where the case $n=0$ is replaced by the limiting case $n\to 0$ and the orthogonality conditions (B8) become:

$$\int_{\alpha}^{\pi-\alpha}T_m(\mathrm{X})T_n(\mathrm{X})\frac{d\vartheta}{\sqrt{\cos 2\alpha-\cos 2\vartheta}}=\frac{\sec^2\frac{\alpha}{2}}{2\sqrt{2}}K'\cdot\begin{cases}2 & (m=n=0),\\ 1 & (m=n\neq 0),\\ 0 & (m\neq n)\end{cases} \tag{B11}$$

and X is expressed by formula (B9). Note that in (B10) according to (B7)

$$\vartheta=\begin{cases}2\operatorname{arctg}\left(\operatorname{tg}\frac{\alpha}{2}\operatorname{sn}(u,k)\right) & (0<\vartheta<\alpha),\\ 2\operatorname{arctg}\left(\operatorname{ctg}\frac{\alpha}{2}\Big/\operatorname{sn}(u,k)\right) & (\pi-\alpha<\vartheta<\pi)\end{cases}\quad (0<u<K) \tag{B12}$$

where $k=c/d=\operatorname{tg}^2(\alpha/2)\quad(0<\alpha<\pi/2)$ and hence $k'=\sqrt{1-\operatorname{tg}^4(\alpha/2)}$.

Spectral and related to them integral relations for the integral operator in (5) can be obtained directly from the relations (B9) – (B10) by returning to the previous variables (9) and taking into account formulas (B12). However, the relations (B9) - (B10) in the trigonometric forms are somewhat simpler.

We also transform the integral relations (B2). We get

$$\frac{1}{\pi}\int_{c}^{d}\ln\frac{y+\mathrm{v}}{|y-\mathrm{v}|}\frac{T_n(\mathrm{V})\,d\mathrm{v}}{\sqrt{(d^2-\mathrm{v}^2)(\mathrm{v}^2-c^2)}}=\frac{(-1)^n K'\operatorname{sh}(\pi nu/K')}{\pi nd\operatorname{ch}(\pi nK/K')}$$

$$\left(n=0,1,2,...,\ 0<u<K,\ y>d\right).$$

After differentiation of both sides of this relation with respect to y and taking into account that according to (B3b) in case $y>d$, we will come to equality

$$\frac{1}{\pi}\int_{c}^{d}\frac{T_n(\mathrm{V})2\mathrm{v}d\mathrm{v}}{(\mathrm{v}^2-y^2)\sqrt{(d^2-\mathrm{v}^2)(\mathrm{v}^2-c^2)}}=(-1)^{n+1}\frac{\operatorname{ch}(\pi nu/K')}{\operatorname{ch}(\pi nK/K')}\frac{1}{\sqrt{(x^2-d^2)(x^2-c^2)}}\quad\left(n=0,1,2,...,\ y>d\right).$$

Introduce new variables ζ,τ:

$$y^2=\zeta+\left(c^2+d^2\right)/2,\quad \mathrm{v}^2=\tau+\left(c^2+d^2\right)/2\quad\left(-b<\zeta,\tau<b;b=\left(d^2-c^2\right)/2\right).$$

After elementary transformations we obtain

$$\frac{1}{\pi}\int_{-b}^{b}\frac{T_n(V)\,d\tau}{(\tau-\zeta)\sqrt{b^2-\tau^2}}=(-1)^{n+1}\frac{\operatorname{ch}(\pi nu/K')}{\operatorname{ch}(\pi nK/K')}\frac{1}{\sqrt{\zeta^2-b^2}}\quad (n=0,1,2,...,\ \zeta>b),\quad \text{(B13)}$$

where according to (B1)

$$V=\cos\Phi,\quad \Phi=\frac{\pi}{K'}\int_{1}^{\gamma(\tau)}\frac{dt}{\sqrt{(t^2-1)(1-k^2t^2)}},\quad \gamma(\tau)=\frac{1}{c\sqrt{2}}\sqrt{2\tau+c^2+d^2}\qquad \text{(B14)}$$

and for variable u $(0<u<K')$ again by (B3b) we have

$$u=\int_{0}^{snu}\frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}=\int_{0}^{\chi(\zeta)}\frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}};\ \chi(\zeta)=\frac{d\sqrt{2}}{\sqrt{2\zeta+c^2+d^2}}.\qquad \text{(B15)}$$

Since here $b=(d^2-c^2)/2=(\operatorname{ctg}^2\alpha/2-\operatorname{tg}^2\alpha/2)/2=2\cos\alpha/\sin^2\alpha$ $(0<\alpha<\pi/2)$, then $0<b<\infty$ and hence in (B13)–(B15), we can formally replace b by a; ζ by x; τ by s and express other parameters in terms of a. Namely, it is easy to find that

$$\left.\begin{aligned}&c=\sqrt{\sqrt{1+a^2}-a},\quad d=\sqrt{a+\sqrt{1+a^2}},\quad k=\sqrt{1+a^2}-a;\ \cos\alpha=\frac{a}{1+\sqrt{1+a^2}};\\&\frac{c^2+d^2}{2}=\sqrt{1+a^2};\ \cos^2\frac{\alpha}{2}=\frac{1}{2a}\left(a-1+\sqrt{1+a^2}\right);\quad \cos^2\alpha=\frac{1}{a^2}\left(2+a^2-2\sqrt{1+a^2}\right).\end{aligned}\right\}\ \text{(B16)}$$

Now taking into account these changes, we rewrite the relationships (B13)–(B15):

$$\left.\begin{aligned}&\frac{1}{\pi}\int_{-a}^{a}\frac{T_n(\mathrm{V})\,ds}{(s-x)\sqrt{a^2-s^2}}=(-1)^{n+1}\frac{\operatorname{ch}(\pi nu/K')}{\operatorname{ch}(\pi nK/K')}\frac{1}{\sqrt{x^2-a^2}}\ \ (n=0,1,2,...;\ x>a),\\&\mathrm{V}=\cos\Phi,\ \ \Phi=\frac{\pi}{K'}\int_{1}^{\gamma(s)}\frac{dt}{\sqrt{(t^2-1)(1-k^2t^2)}},\ \ \gamma(s)=\frac{1}{a}\sqrt{s+\sqrt{1+a^2}},\\&u=\int_{0}^{\chi(x)}\frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}},\quad \chi(x)=\sqrt{\frac{a+\sqrt{1+a^2}}{x+\sqrt{1+a}}},\end{aligned}\right\}\ \text{(B17)}$$

where the value of modulus k is given in (B16).

Let us consider also the limiting case of the discussing problem as $a\to l$. In this case, $\rho=1$ should be substituted into the equations (5) - (7), the relation (8) is excluded from consideration, and the integral equation (10) takes the form

$$\frac{2}{\pi}\int_{0}^{\pi}\ln\left(\sin\left(\frac{\vartheta+\varphi}{2}\right)\Big/\left|\sin\left(\frac{\vartheta-\varphi}{2}\right)\right|\right)\omega_0(\varphi)\,d\varphi=g_0(\vartheta)\quad (0<\vartheta<\pi)\qquad \text{(B18)}$$

The eigenfunctions and eigenvalues of the kernel of the equation (B18) can be easily found using well-known Fourier series [23] (p.52, f-la 1.441.2):

$$\ln\left(1\Big/2\left|\sin\frac{x}{2}\right|\right)=\sum_{n=1}^{\infty}\frac{\cos nx}{n}\left(-2\pi<x<2\pi\right).$$

Whence we get

$$\ln\left(\sin\left(\frac{\vartheta+\varphi}{2}\right)\Big/\left|\sin\left(\frac{\vartheta-\varphi}{2}\right)\right|\right)=2\sum_{n=1}^{\infty}\frac{\sin n\vartheta\sin n\varphi}{n}\qquad\left(0<\vartheta,\ \varphi<\pi\right).$$

Now if we multiply both sides of this expansion by $\sin m\varphi\ \ (m=1,2,...)$ and integrate the resulting equality with respect to φ over the interval $(0,\pi)$, we come to the following spectral relations $(m\to n)$

$$\frac{1}{\pi}\int_0^{\pi}\ln\left(\frac{\sin(\vartheta+\varphi)}{2}\Big/\left|\sin\left(\frac{\vartheta-\varphi}{2}\right)\right|\right)\sin n\varphi d\varphi=\frac{1}{n}\sin n\vartheta\quad\left(n=1,2,...,0<\vartheta<\pi\right).$$

This relation in the former variables (9) with $\rho=1$ takes the form [29]

$$\frac{1}{2\pi}\int_{-1}^{1}\ln\frac{1-\xi\eta+\sqrt{\left(1-\xi^2\right)\left(1-\eta^2\right)}}{1-\xi\eta-\sqrt{\left(1-\xi^2\right)\left(1-\eta^2\right)}}U_{n-1}(\eta)d\eta=\frac{1}{n}\sqrt{1-\xi^2}U_{n-1}(\xi)\ \left(-1<\xi<1;\ n=1,2,...\right)\tag{B19}$$

Appendix C. To analytically explore the behavior of K_I^0 as $\chi\to1$ or $\rho\to1$, we substitute in the formula (7) (as in section 4)

$p_0(\xi)=\Omega(\xi,\rho)\Big/\sqrt{1-\xi^2}\quad(-1<\xi<1)$,

where $\Omega(\xi,\rho)$ is the even Hölder function on the interval $[-1,1]$. Then we transform this formula as follows:

$$K_I^0=\int_{-1}^{1}\sqrt{\frac{\rho+\eta}{\rho-\eta}}\frac{\Omega(\eta,\rho)d\eta}{\sqrt{1-\eta^2}}=\int_{-1}^{1}\sqrt{\frac{\rho+\eta}{\rho-\eta}}\frac{A\eta+B}{\sqrt{1-\eta^2}}d\eta+\int_{-1}^{1}\sqrt{\frac{\rho+\eta}{\rho-\eta}}\frac{\Omega(\eta,\rho)-A\eta-B}{\sqrt{1-\eta^2}}d\eta$$

where A and B are not yet known constants. Set then $\Omega_0(\eta,\rho)=\Omega(\eta,\rho)-A\eta-B$ $(-1\le\eta\le1)$ and determine the constants A and B from the conditions $\Omega_0(\pm1,\rho)=0$. As a result,

$A=\left[\Omega_0(1,\rho)-\Omega_0(-1,\rho)\right]\big/2;\quad B=\left[\Omega_0(1,\rho)+\Omega_0(-1,\rho)\right]\big/2$

and due to the parity of the function $\Omega_0(\eta,\rho)\quad A=0,\ B\neq0$. Then we can write

$$K_I^0=B\int_{-1}^{1}\sqrt{\frac{\rho+\eta}{\rho-\eta}}\frac{d\eta}{\sqrt{1-\eta^2}}+\int_{-1}^{1}\sqrt{\frac{\rho+\eta}{\rho-\eta}}\frac{\Omega_0(\eta,\rho)}{\sqrt{1-\eta^2}}d\eta,\quad\Omega_0(\eta,\rho)=\Omega(\eta,\rho)-B.\tag{C1}$$

Since $\Omega_0(\pm1,\rho)=0$, the second integral in (65a) is limited when $\rho\to1$, and the first integral is easily calculated. Namely,

$$I(\rho)=\int_{-1}^{1}\sqrt{\frac{\rho+\eta}{\rho-\eta}}\frac{d\eta}{\sqrt{1-\eta^2}}=\int_{-1}^{1}\frac{(\rho+\eta)d\eta}{\sqrt{\left(1-\eta^2\right)\left(\rho^2-\eta^2\right)}}=\rho\int_{-1}^{1}\frac{d\eta}{\sqrt{\left(1-\eta^2\right)\left(\rho^2-\eta^2\right)}}=2K(\chi)\ (\chi=1/\rho)\tag{C2}$$

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