ՀԱՅԱՍՏԱՆԻ ԳԻՏՈՒԹՅՈՒՆՆԵՐԻ ԱԶԳԱՅԻՆ ԱԿԱԴԵՄԻԱՅԻ ՏԵՂԵԿԱԳԻՐ
ИЗВЕСТИЯ НАЦИОНАЛЬНОЙ АКАДЕМИИ НАУК АРМЕНИИ

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AVERAGED CONTROLLABILITY OF EULER-BERNOULLI BEAMS WITH RANDOM MATERIAL CHARACTERISTICS: THE GREEN'S FUNCTION APPROACH

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Keywords: averaged dynamics, mathematical expectation, parameter-dependent systems

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Усреднённая управляемость балок Эйлера-Бернулли со случайными характеристиками материала: метод функции Грина

Ключевые слова: усреднённая динамика, математическое ожидание, системы, зависящие от параметра

Исследуются точная и приближённая усреднённая управляемость балки Эйлера-Бернулли со случайными характеристиками (жёсткость на изгиб и плотность). Рассматриваются случаи, когда материальные характеристиками являются равномерно и нормально распределённые случайные величины. Задача заключается в определении допустимых управлений, точно или приближённо обеспечивающих требуемое усреднё,нное состояние балки за заданное конечное время. Представив общее решение уравнения изгиба балки с помощью функции Грина, получаются явные представления для прогиба и скорости точек балки, тем самым, упрощая анализ усреднённой управляемости балки. Выводятся необходимые и достаточные условия точной усреднённой управляемости балки, а также достаточные условия приближённой усреднённой управляемости балки. Основные результаты подтверждаются численными расчётами. Функции Грина для уравнения изгиба балки при различных граничных условиях и соответствующие усреднённые решения приводятся в приложениях.

Ավետիսյան Ա.Ս., Խուրշուդյան Ա.Ժ.

Նյութի պատահական բնութագրիչներով էյլեր-Բերնուլիի հեծանների միջինացված ղեկավարելիությունը. Գրինի ֆունկցիայի եղանակը

Հիմնաբառեր. միջինացված դինամիկա, մաթեմատիկական սպասում, պարամետրից կախված համակարգեր

Հետազոտվում է նյութի պատահական բնութագրիչներով (ծռման կոշտություն և խտություն) Էյլեր-Բերնուլիի հեծանի միջինացված ճշգրիտ և գրեթե ղեկավարելիությունը: Դիտարկվում են հեծանի նյութի բնութագրիչների հավասարաչափ և նորմալ բաշխված պատահական մեծություն լինելու դեպքերը: Որոնվում են տրված վերջավոր ժամանակում հեծանի պահանջվող վիճակն (ճշգրիտ և մոտավոր) ապահովող թույլատրելի ղեկավարումները: Միջինացված ղեկավարելիության հետազոտումը պարզեցնելու նպատակով՝ հեծանի ծռման հավասարման ընդհանուր լուծումը ներկայացվում է Գրինի ֆունկցիայի միջոցով: Հեծանի միջինացված ճշգրիտ ղեկավարելիության համար ստացվում են անհրաժեշտ և բավարար, իսկ միջինացված գրեթե ղեկավարելիության համար՝ բավարար պայմաններ: Հետազոտության արդյունքը հաստատվում է օրինակներով: Հավելվածներում բերվում են հեծանի ծռման հավասարման Գրինի ֆունկցիան՝ տարատեսակ եզրային պայմանների համար, ինչպես նաև համապատասխան միջինացված լուծումները:

We examine the Euler-Bernoulli beam with random material characteristics (bending stiffness and mass per unit length) on exact and approximate averaged controllability. Cases when the material characteristics are standard normally and uniformly distributed random variables are considered. The problem is in an appropriate choice of admissible controls providing a required averaged state of the beam (exactly or approximately) within a desired amount of time. Representing the general solution of the beam equation in terms of the Green's function, it becomes possible to derive explicit closed-form representations for the averaged deflection and velocity of the beam. This makes controllability analysis a matter of straightforward computations. Specifically, necessary and sufficient conditions for the exact averaged controllability, as well as sufficient conditions for the approximate averaged

controllability are derived with respect to admissible controls. Numerical analysis allows to make a sensible understanding of theoretical derivations. Green's functions for the main types of boundary conditions and closed-form representation of the averaged dynamics are defined in appendices.

Introduction

The dynamics described by random state constraints (e.g., differential equations accompanied by initial/boundary conditions), generally, is a random function. Therefore, if such a dynamics is controlled, then the control function also must be random. However, dealing with controllability analysis, control function must not contain any randomness. A way of overcoming such situations for systems containing random parameters has been suggested by the prominent mathematician Enrique Zuazua in [1] where a general theory of controllability of finite-dimensional system has been developed. The compromise is found in a smart way by controlling the averaged dynamics or the mathematical expectation of the dynamics over all possible values of the random parameters. This new type of controllability is called averaged controllability. A general theory for infinite-dimensional or distributed parameter system is currently under development by Zuazua and colleagues. See [2-5] for some of existing contributions. See also [4] for a handful of open problems related to controllability of random evolution equations.

Suppose that the controlled state of a dynamic system is described by a function (for the sake of simplicity, we restrict ourselves by the scalar case) $w: \mathcal{U} \times \mathbb{R}^n \times \mathbb{R}^+ \times \Omega \to \mathcal{R}$ where $\Omega \in \mathbb{R}^m$ is the domain of random parameters contained in the state constraints on w (imagine, e.g., a differential equation with initial and boundary conditions), and $\mathcal{U}$ is the set of admissible controls. Then, instead of the usual controllability residue [6]

$$\mathcal{R}_T(u) = \left\| w(u, x, T; \boldsymbol{\omega}) - w_T \right\|_{\mathbf{W}_T},$$

where T is the control time, $\boldsymbol{\omega} \in \Omega$ is the vector of random variables, w_T is the desired terminal state and $\mathbf{W}_T$ is the space of terminal states, Zuazua suggests to consider the averaged residue [3]

$$\mathcal{R}_T^{av}(u) = \left\| \int_\Omega w(u, x, T; \boldsymbol{\omega}) \mathrm{d}\mathbb{P}(\boldsymbol{\omega}) - w_T \right\|_{\mathbf{W}_T},$$

where the integral of w over Ω would be the averaged state or the mathematical expectation. After computing the expectation and substituting it into $\mathcal{R}_T^{av}$, it will be guaranteed that the control u does no longer need to be dependent on $\boldsymbol{\omega}$.

At this, following to [3], we distinguish two concrete types of averaged controllability. If for any initial and desired states, control time T, $\mathcal{R}_T^{av}(u) = 0$ for a $u \in \mathcal{U}$, then the system is exactly averaged controllable. If for any initial and desired states, control time T and a desired accuracy $\varepsilon > 0$, $\mathcal{R}_T^{av}(u) < \varepsilon$ for a $u \in \mathcal{U}$, then the system is approximately averaged controllable. Null-averaged (exact and approximate) controllable systems are defined analogously. Admissible controls providing exact (approximate) averaged controllability, are called exactly (approximately) resolving average controls.

In this paper, we study the averaged controllability of Euler-Bernoulli beam with random parameters. The cases of uniformly and normally distributed random variables are considered. Representing the general solution of the Euler-Bernoulli beam equation in terms of corresponding Green's function and making use of the Green's function approach [6, 7], we derive exact and approximate averaged controllability constraints. Numerical analysis reveals non-triviality of theoretical derivations. At this, it is worth mentioning that the averaging process considered in this paper has nothing in common with the averaging of material characteristics widely applied in mechanics of inhomogeneous materials (see, e.g., [8]).

Note that the averaged controllability of Euler-Bernoulli beams has been considered in a recent paper [9]. However, we would like to point out two principal differences between that and the current papers. First of all, in [9] only the flexural stiffness E is considered as a random variable, where E is the Young's modulus, I is the moment of inertia of the cross section of the beam, while in this paper, besides EI, another important characteristic of the beam - μ, the mass per unit length, is considered as a random variable. This becomes important especially when dealing with problems for beams made of specific material with distinct Young's modulus and density. The second distinctive feature is that the average controllability analysis based on the Green's function approach is quite straightforward and it is easier to apply in particular cases, since the Green's function of the beam equations with various boundary conditions has been found explicitly.

1. Governing equation and its Green's function solution

We consider a Euler-Bernoulli beam subject to a distributed control influence. Then, the vertical displacement of the beam is determined from the fourth-order PDE (all variables and quantities are dimensionless)

$$\frac{\omega_1}{\omega_2}\frac{\partial^4 w}{\partial x^4}+\frac{\partial^2 w}{\partial t^2}=\frac{1}{\omega_2}u(t)v(x),\ 0<x<l,\ t>0, \tag{1.1}$$

where $\omega_1 = EJ > 0$ and $\omega_2 = \mu > 0$ are the beam flexural stiffness and mass per unit length, respectively. Control influence is described by u with distribution v. By a proper choice of v, boundary controls can also be considered.

Hereinafter, we limit the consideration by the case when both ω_1 and ω_2 are either standard normally or uniformly distributed independent random variables. Then, their probability density functions are given by

$$\rho(\omega_1,\omega_2)=\frac{1}{2\pi}\exp\left[-\frac{\omega_1^2+\omega_2^2}{2}\right] \tag{1.2}$$

or

$$\rho(\omega_1,\omega_2)=\frac{1}{\mu(\Omega)}\chi_\Omega(\omega_1,\omega_2), \tag{1.3}$$

where χ_Ω is the indicator function, and $\mu(\Omega)$ is the measure of Ω.

Let at $t=0$ the beam is in equilibrium. Then, the general solution of (1.1) can be represented in terms of Green's function [10]

$$w(x,t;\omega_1,\omega_2)=\frac{1}{\omega_2}\int_0^l\int_0^t G(x,\xi,t-\tau;\omega_1,\omega_2)u(\tau)v(\xi)\,\mathrm{d}\tau\mathrm{d}\xi, \tag{1.4}$$

$$G(x,\xi,t;\omega_1,\omega_2)=\sum_{n=1}^{\infty}\frac{1}{\lambda_n^2\|\varphi_n\|^2}\varphi_n(x)\varphi_n(\xi)\sin\left(\lambda_n^2\sqrt{\frac{\omega_1}{\omega_2}}t\right).$$

For determination of λ_n and φ_n for multiple boundary conditions, see Appendix 1. Therefore, (1.4) can be rewritten as follows:

$$w(x,t;\omega_1,\omega_2)=\frac{1}{\omega_2}\sum_{n=1}^{\infty}\frac{\alpha_n}{\lambda_n^2\|\varphi_n\|^2}\varphi_n(x)\int_0^t u(\tau)\sin\left[\lambda_n^2\sqrt{\frac{\omega_1}{\omega_2}}(t-\tau)\right]\mathrm{d}\tau, \tag{1.5}$$

$$\alpha_n=\int_0^l\varphi_n(\xi)v(\xi)\,\mathrm{d}\xi.$$

The particle velocity of the beam is determined by differentiating (1.5}) w.r.t. t:

$$\frac{\partial w}{\partial t}(x,t;\omega_1,\omega_2)=\frac{\sqrt{\omega_1}}{\omega_2^{3/2}}\sum_{n=1}^{\infty}\frac{\alpha_n}{\|\varphi_n\|^2}\varphi_n(x)\int_0^t u(\tau)\cos\left[\lambda_n^2\sqrt{\frac{\omega_1}{\omega_2}}(t-\tau)\right]\mathrm{d}\tau. \tag{1.6}$$

2. Averaged controllability via the Green's function approach

The averaged controllability problem for the beam can now be formulated as follows. Given any $w_T, w_{T1}\in L^2[0,l]$ and control time T, find control functions $u\in\mathcal{U}=\{u\in L^2[0,T],\ \mathrm{supp}(u)\subseteq[0,T]\}$ such that the averaged residue

$$\mathcal{R}_T^{av}(u)=\left\|\mathbb{M}_T^{\Omega}[w]-w_T\right\|_{L^2[0,l]}^2+\left\|\mathbb{M}_T^{\Omega}\left[\frac{\partial w}{\partial t}\right]-w_{T1}\right\|_{L^2[0,l]}^2 \tag{2.1}$$

satisfies $\mathcal{R}_T^{av}(u)=0$ or $\mathcal{R}_T^{av}(u)<\varepsilon$ with a desired accuracy ε. Here,

$$\mathbb{M}_T^{\Omega}[w]=\int_{\Omega}w(x,T;\omega_1,\omega_2)\,\mathrm{d}\mathbb{P}(\omega_1,\omega_2)=\int_{\Omega}w(x,T;\omega_1,\omega_2)\rho(\omega_1,\omega_2)\,\mathrm{d}\Omega$$

is the mathematical expectation of w over Ω at $t=T$.

Straightforward calculations provide

$$\mathbb{M}_T^{\Omega}[w]=\sum_{n=1}^{\infty}\frac{\alpha_n}{\lambda_n^2\|\varphi_n\|_{L^2[0,l]}^2}\varphi_n(x)\int_0^T\Psi_n(T-\tau)u(\tau)\,\mathrm{d}\tau:=\sum_{n=1}^{\infty}\beta_n(T,u)\varphi_n(x),$$

$$\mathbb{M}_T^{\Omega}\left[\frac{\partial w}{\partial t}\right]=\sum_{n=1}^{\infty}\frac{\alpha_n}{\|\varphi_n\|_{L^2[0,l]}^2}\varphi_n(x)\int_0^T\Psi_{n1}(T-\tau)u(\tau)\,\mathrm{d}\tau:=\sum_{n=1}^{\infty}\beta_{n1}(T,u)\varphi_n(x),$$

where

$$\beta_n(T,u)=\frac{\alpha_n}{\lambda_n^2\|\varphi_n\|^2_{L^2[0,l]}}\int_0^T \Psi_n(T-\tau)u(\tau)\,\mathrm{d}\tau,$$

$$\Psi_n(t)=\int_\Omega \frac{1}{\omega_2}\sin\left[\lambda_n^2\sqrt{\frac{\omega_1}{\omega_2}}t\right]\rho(\omega_1,\omega_2)\,\mathrm{d}\Omega,$$

$$\beta_{n1}(T,u)=\frac{\alpha_n}{\|\varphi_n\|^2_{L^2[0,l]}}\int_0^T \Psi_{n1}(T-\tau)u(\tau)\,\mathrm{d}\tau,$$

$$\Psi_{n1}(t)=\int_\Omega \frac{\sqrt{\omega_1}}{\omega_2^{3/2}}\cos\left[\lambda_n^2\sqrt{\frac{\omega_1}{\omega_2}}t\right]\rho(\omega_1,\omega_2)\,\mathrm{d}\Omega.$$

Furthermore, note that

$$\left\|\mathbb{M}_T^\Omega[w]-w_T\right\|^2_{L^2[0,l]}=\sum_{n=1}^\infty\left(\beta_n(T,u)-w_{Tn}\right)^2\|\varphi_n\|^2_{L^2[0,l]},$$

reducing (2.1) to

$$\mathcal{R}_T^{av}(u)=\sum_{n=1}^\infty\left[\left(\beta_n(T,u)-w_{Tn}\right)^2+\left(\beta_{n1}(T,u)-w_{T1n}\right)^2\right]\|\varphi_n\|^2_{L^2[0,l]}, \qquad (2.2)$$

where w_{Tn} and w_{T1n} are the expansion coefficients of w_T and w_T^1 into series of $\{\varphi_n\}_{n=1}^\infty$

2.1. Exact controllability

Then, from (2.2) we straightforwardly obtain the following result.

Theorem 1. For the beam exact averaged controllability it is necessary and sufficient that for given T,

$$\begin{cases}\beta_n(T,u)-w_{Tn}=0,\\ \beta_{n1}(T,u)-w_{T1n}=0,\end{cases} \quad n=1,2,\dots \qquad (2.3)$$

for $u\in\mathcal{U}$.

Remark 1. Note that system (2.3) is linear in u. Therefore, the set of exactly resolving averaged controls can be described by solving (2.3) as a infinite dimensional linear problem of moments, L^p – optimal solution of which for $1\le p\le\infty$ has been explicitly derived in [11]. The heuristic method [12] can be applied, too.

2.2. Approximate controllability

Making use of the triangle inequality, for $\mathcal{R}_T^{av}$ we obtain the following estimate:

$$\mathcal{R}_T^{av} \le \sum_{n=1}^{\infty}\left[\beta_n^2(T,u)+\beta_{n1}^2(T,u)+w_{Tn}^2+w_{T1n}^2\right]\|\varphi_n\|_{L^2[0,l]}^2 .$$

Then, the following assertion holds.

Theorem 2. If for given T,

$$\sum_{n=1}^{\infty}\left[\beta_n^2(T,u)+\beta_{n1}^2(T,u)+w_{Tn}^2+w_{T1n}^2\right]\|\varphi_n\|_{L^2[0,l]}^2 \le \varepsilon \tag{2.4}$$

for $u \in \mathcal{U}$, then the beam is approximately averaged controllable.

Remark 2. Note that (2.4) makes sense only when

$$\tilde{\varepsilon} := \varepsilon - \sum_{n=1}^{\infty}\left[w_{Tn}^2+w_{T1n}^2\right]\|\varphi_n\|_{L^2[0,l]}^2 \ge 0. \tag{2.5}$$

Moreover, making use of the Cauchy-Schwartz inequality implying

$$\left[\int_0^T \Psi_n(T-\tau)u(\tau)\,\mathrm{d}\tau\right]^2 \le \|u\|_{L^2[0,T]}^2\|\Psi_n\|_{L^2[0,T]}^2 ,$$

we obtain

$$\beta_n^2(T,u)+\beta_{n1}^2(T,u)=\frac{\alpha_n^2}{\lambda_n^4\|\varphi_n\|_{L^2[0,l]}^4}\left[\int_0^T \Psi_n(T-\tau)u(\tau)\,\mathrm{d}\tau\right]^2+$$

$$+\frac{\alpha_n^2}{\|\varphi_n\|_{L^2[0,l]}^4}\left[\int_0^T \Psi_{n1}(T-\tau)u(\tau)\,\mathrm{d}\tau\right]^2 \le \frac{\alpha_n^2\|u\|_{L^2[0,T]}^2}{\lambda_n^4\|\varphi_n\|_{L^2[0,l]}^4}\left[\|\Psi_n\|_{L^2[0,T]}^2+\lambda_n^4\|\Psi_{n1}\|_{L^2[0,T]}^2\right].$$

Therefore, the following assertion holds.

Corollary 1. Assume that the desired state w_T, w_{T1} is constrained by inequality (2.5). If for given T,

$$\sigma(T)=\sum_{n=1}^{\infty}\frac{\alpha_n^2}{\lambda_n^4\|\varphi_n\|_{L^2[0,l]}^4}\left[\|\Psi_n\|_{L^2[0,T]}^2+\lambda_n^4\|\Psi_{n1}\|_{L^2[0,T]}^2\right]\ne 0 \tag{2.6}$$

is finite and

$$\|u\|_{L^2[0,T]}^2\sigma(T)<\tilde{\varepsilon}, \tag{2.7}$$

then the Euler-Bernoulli beam is approximate controllable.

Remark 3. Moreover, if the conditions of Corollary 1 hold, then (2.7) defines a subset of approximately resolving controls. Namely, any admissible control $u \in \mathcal{U}$ with

$$\|u\|_{L^2[0,T]}^2 < \frac{\tilde{\varepsilon}}{\sigma(T)}$$

is an approximately resolving control.

Remark 4. When $\Omega = \{\omega_1, \omega_2 \in \mathbb{R}^+, \omega_{i0} \le \omega_i \le \omega_{i1}, i = 1, 2\}$ is a rectangle, using the boundedness of Ψ_n and Ψ_{n1} on $[0,T]$ for all $n = 1, 2, \ldots$, we see that $\|\Psi_n\|_{L^2[0,T]}$, $\|\Psi_{n1}\|_{L^2[0,T]}$ are finite, so that σ is finite for $T \ge t_0 > 0$. Therefore, as soon as $\Psi_n \not\equiv 0$ or $\Psi_{n1} \not\equiv 0$ for at least one n, then (2.6) holds.

3. Numerical analysis

In this section, we carry out a numerical analysis of a particular case when Ω is a rectangle. For the sake of simplicity, as well as in order to be eligible to involve the Euler-Bernoulli assumptions, we limit the consideration only by metals (see Table 1). Assume that the beam is of unit length, has a square cross section of side $h = l/100$, and is simply supported (for explicit representation of the corresponding Green's function, see Appendix 1, case 2). The beam is subjected to a control concentrated at the mid-point of the beam, i.e., $v(x) = \delta(x - 1/2)$, where δ is the Dirac function.

Table 1. Young's modulus and densities of some metals

Metal	E [GPa]	ρ [kg/m^3]
Aluminum	10	2800
Cast iron	13.4	7300
Titanium	15.5	4500
Al-Bronze	17	8200
Monel 400	26	8600
Steel	29.2	7850
Cr-Mo Steel	31.7	8000

Based on the values presented in Table 1, we compute $\omega_{10} = 25/3$ (Aluminum), $\omega_{11} = 26.4167$ (Cr-Mo Steel), $\omega_{20} = 7/25$ (Aluminum), $\omega_{21} = 17/20$ (Monel).

3.1. Uniformly distributed random variables

First, let us consider the case of uniformly distributed independent random variables with (1.2). Numerical analysis shows that, in this case, both Ψ_n and Ψ_{n1} are bounded functions of t and tend to 0 very fast as t increases (see Fig.1). For explicit representation of Ψ_n and Ψ_{n1}, see Appendix 2.

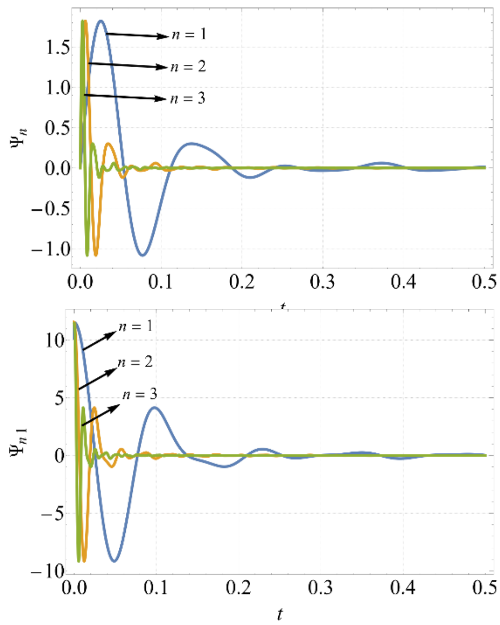

Fig.1. Plots of Ψ_n and Ψ_{n1} against t for $n=1,2,3$: the case of uniformly distributed random variables

Evaluation of σ as in (2.6) shows that as T increases, $\sigma(T)$ increases from 0 and approaches the value of ≈ 3.13125 (see Fig. 2). Therefore, the first condition of Corollary 1 holds. Since in this case, $2\|\varphi_n\|^2_{L^2[0,l]}=1$, then (2.5) holds with $\varepsilon=10^{-4}$, e.g., for $w_T=a\sin(\pi x)$, $w_{T1}\equiv 0$ in $[0,1]$ with $a\le 10^{-2}$. Implying Corollary 1, we see that admissible controls with $\|u\|^2_{L^2[0,T]}<1.66\cdot 10^{-5}$ provide approximate averaged controllability of the beam.

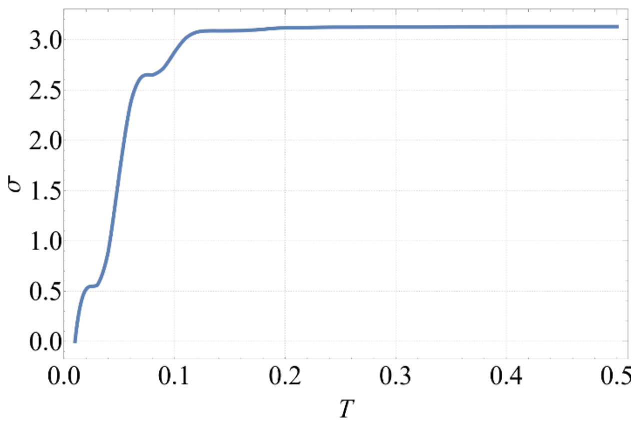

Fig. 2. Plot of σ against T : the case of uniformly distributed random variables

3.2. Normally distributed random variables

Consider now the case when ω_1 and ω_2 are normally distributed independent random variables with (1.2). It seems that it is impossible to express Ψ_n and Ψ_{1n} in an explicit form. Nonetheless, numerical analysis shows that $\left|\Psi_n\right| \leq 3 \cdot 10^{-18}$ and $\left|\Psi_{1n}\right| \leq 8 \cdot 10^{-16}$ on $\left[0, T\right]$ for all $n = 1, 2, \ldots,$ providing in (2.6), $\sigma \leq 10^{-34}$. Evidently, in this case Corollary 1 does not hold, and for the establishment of approximate averaged controllability of the heat equation, inequality (2.4) must be evaluated.

Conclusions

Using the Green's function approach, necessary and sufficient conditions for exact averaged controllability, as well as sufficient conditions for approximate averaged controllability of a Euler-Bernoulli beam with random material characteristics subjected to multiple boundary conditions are obtained in this paper. At this, both cases of standard normally and uniformly distributed independent random variables are covered. The averaged dynamics of the beam is represented in terms of the random characteristics explicitly, which simplifies the controllability analysis significantly. The determination of exactly resolving average controls is reduced to an infinite-dimensional linear problem of moments, the general solution of which in L^p , $1 \leq p \leq \infty$, is known. A simple inequality on the L^2 -norm of the approximately resolving average controls is derived. Numerical analysis reveals the efficacy in the sense of computational complexity of the derived constraints.

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REFERENCES

1. Zuazua E., Averaged control. //Automatica, vol. 50, №12, pp. 3077-3087. (2014).
2. Lazar M., Zuazua E., Averaged control and observation of parameter depending wave equations. C. R. Math. Acad. Sci. Paris, vol. 352, №6, pp. 497-502 (2014).
3. Lü Q., Zuazua E., Averaged controllability for random evolution Partial Differential Equations. //Journal de Mathématiques Pures et Appliquées, vol. 1015, №3, pp. 367-414 (2016).
4. Lü Q., Zuazua E., Averaged controllability for parameter-dependent evolution partial differential equations. Open Problems in Systems and Control, IMA-Minneapolis, pp. 1-4 (2017).
5. Lohéac J., Zuazua E., Averaged controllability of parameter dependent conservative semigroups. //Journal of Differential Equations, vol. 262, № 3, pp.1540-1574 (2017).
6. Avetisyan A.S., Khurshudyan As. Zh., Controllability of Dynamic Systems: The Green's Function Approach. Cambridge Scholars Publishing, Cambridge, 2018.
7. Avetisyan A.S., Khurshudyan As.Zh., Green‘s function approach in approximate controllability problems. Mechanics //Proceedings of National Academy of Sciences of Armenia, vol. 69, issue 2, pp.3-22 (2016).
8. Bakhvalov N.S., Panasenko G., Homogenisation: Averaging Processes in Periodic Media. Kluwer Academic Publishers, Dordrecht, 1989.

9. Marin F.J., Martines-Frutos J., Periago F., Robust averaged control of vibrations for the Bernoulli–Euler beam equation. //Journal of Optimization Theory & Applications, vol.174, pp. 428-454 (2017).
10. Polyanin A.D., Nazaikinski V.E., Handbook of Linear Partial Differential Equations. 2nd edition. Chapman & Hall/CRC Press, Boca Raton, 2016.
11. Khurshudyan As.Zh., Structural Optimization and Control of Vibrations of Deformable Systems with Variable Parameters, PhD dissertation. Institute of Mechanics, National Academy of Sciences of Armenia, 2016.
12. Khurshudyan As.Zh., Heuristic determination of resolving controls for exact and approximate controllability of nonlinear dynamic systems. Mathematical Problems in Engineering, Article ID 9496371, 16 pages (2018).

Appendix 1

In this appendix we define the eigenfunctions φ_n and eigenvalues λ_n for five commonly considered boundary conditions. Below,

$$K_{1n,3n}(x)=\sinh(\lambda_n x)\mp\sin(\lambda_n x),\ K_{2n,4n}(x)=\cosh(\lambda_n x)\mp\cos(\lambda_n x),$$

are the Krylov functions.

1. Both ends are clamped:

$$w=\frac{\partial w}{\partial x}=0 \text{ at } x=0 \text{ and } x=l.$$

Then,

$$\frac{1}{\lambda_n^2\|\varphi_n\|^2}=\frac{4}{l}\frac{\lambda_n^2}{\left[\varphi_{n''}(l)\right]^2},\ \varphi_n(x)=K_{1n}(l)K_{2n}(x)-K_{2n}(l)K_{1n}(x),$$

λ_n are the positive roots of the transcendental equation $\cosh(\lambda l)\cos(\lambda l)=1$.

2. Both ends are simply supported:

$$w=\frac{\partial^2 w}{\partial x^2}=0 \text{ at } x=0 \text{ and } x=l.$$

Then,

$$\frac{1}{\lambda_n^2\|\varphi_n\|^2}=\frac{2l}{\pi^2 n^2},\ \varphi_n(x)=\sin(\lambda_n x),\ \lambda_n=\frac{\pi n}{l}.$$

3. One end is clamped, one end is simply supported:

$$w=\frac{\partial w}{\partial x}=0 \text{ at } x=0 \text{ and } w=\frac{\partial^2 w}{\partial x^2}=0 \text{ at } x=l.$$

Then,

$$\frac{1}{\lambda_n^2 \|\varphi_n\|^2} = \frac{2}{l} \frac{\lambda_n^2}{|\varphi_{n'}(l)\varphi_{n''}(l)|}, \ \varphi_n(x) = K_{1n}(l)K_{2n}(x) - K_{2n}(l)K_{1n}(x),$$

λ_n are the positive roots of the transcendental equation $\tan(\lambda l) - \tanh(\lambda l) = 0$;

4. One end is clamped, one end is free:

$$w = \frac{\partial w}{\partial x} = 0 \text{ at } x = 0 \text{ and } w = \frac{\partial^3 w}{\partial x^3} = 0 \text{ at } x = l.$$

Then,

$$\frac{1}{\lambda_n^2 \|\varphi_n\|^2} = \frac{4}{l} \frac{1}{\lambda_n^2 \varphi_n^2(l)}, \ \varphi_n(x) = K_{3n}(l)K_{2n}(x) - K_{4n}(l)K_{1n}(x),$$

λ_n are the positive roots of the transcendental equation $\cosh(\lambda l)\cos(\lambda l) = -1$;

5. One end is simply supported, one end is free:

$$w = \frac{\partial^2 w}{\partial x^2} = 0 \text{ at } x = 0 \text{ and } w = \frac{\partial^3 w}{\partial x^3} = 0 \text{ at } x = l.$$

Then,

$$\frac{1}{\lambda_n^2 \|\varphi_n\|^2} = \frac{4}{l} \frac{1}{\lambda_n^2 \varphi_n^2(l)}, \ \varphi_n(x) = \sin(\lambda_n l)\sinh(\lambda_n x) + \sinh(\lambda_n l)\sin(\lambda_n x),$$

λ_n are the positive roots of the transcendental equation $\tan(\lambda l) - \tanh(\lambda l) = 0$;

Appendix 2

In this appendix, we compute functions Ψ_n and Ψ_{n1} explicitly when ω_1 and ω_2 are uniformly distributed random variables.

Let $\Omega = \{\omega_1, \omega_2 \in \mathbb{R}^+, \omega_{i0} \le \omega_i \le \omega_{i1}, i = 1, 2\}$ be a rectangle. Then, the joint probability density function of ω_1 and ω_2 reads as

$$\rho(\omega_1, \omega_2) = \frac{1}{\mu(\Omega)} \chi_\Omega(\omega_1, \omega_2),$$

where χ_Ω is the indicator function, and $\mu(\Omega) = (\omega_{11} - \omega_{10})(\omega_{11} - \omega_{10})$ is the area of Ω. Then,

$$\Psi_n(t)=\frac{2}{\mu(\Omega)\lambda_n^4 t^2}\left[\omega_{20}S_{1n}(t)-\omega_{21}S_{2n}(t)\right]-\frac{2}{\mu(\Omega)\lambda_n^2 t}\left[C_{1n}(t)-C_{2n}(t)\right]-$$

$$-\frac{2}{\mu(\Omega)\lambda_n^2 t}\left[\omega_{10}S_{3n}(t)-\omega_{11}S_{4n}(t)\right],$$

$$\Psi_{n1}(t)=-\frac{4}{\mu(\Omega)\lambda_n^6 t^3}\left[\omega_{20}S_{1n}(t)-\omega_{21}S_{2n}(t)\right]+\frac{4}{\mu(\Omega)\lambda_n^4 t^2}\left[C_{1n}(t)-C_{2n}(t)\right],$$

where

$$S_{1n}(t)=\sin\left[\lambda_n^2\sqrt{\frac{\omega_{10}}{\omega_{20}}t}\right]-\sin\left[\lambda_n^2\sqrt{\frac{\omega_{11}}{\omega_{20}}t}\right],$$

$$S_{2n}(t)=\sin\left[\lambda_n^2\sqrt{\frac{\omega_{10}}{\omega_{21}}t}\right]-\sin\left[\lambda_n^2\sqrt{\frac{\omega_{11}}{\omega_{21}}t}\right],$$

$$C_{1n}(t)=\sqrt{\omega_{10}\omega_{20}}\cos\left[\lambda_n^2\sqrt{\frac{\omega_{10}}{\omega_{20}}t}\right]-\sqrt{\omega_{11}\omega_{20}}\cos\left[\lambda_n^2\sqrt{\frac{\omega_{11}}{\omega_{20}}t}\right],$$

$$C_{2n}(t)=\sqrt{\omega_{10}\omega_{21}}\cos\left[\lambda_n^2\sqrt{\frac{\omega_{10}}{\omega_{21}}t}\right]-\sqrt{\omega_{11}\omega_{21}}\cos\left[\lambda_n^2\sqrt{\frac{\omega_{11}}{\omega_{21}}t}\right],$$

$$S_{3n}(t)=\mathrm{Si}\left[\lambda_n^2\sqrt{\frac{\omega_{10}}{\omega_{20}}t}\right]-\mathrm{Si}\left[\lambda_n^2\sqrt{\frac{\omega_{10}}{\omega_{21}}t}\right],$$

$$S_{4n}(t)=\mathrm{Si}\left[\lambda_n^2\sqrt{\frac{\omega_{11}}{\omega_{20}}t}\right]-\mathrm{Si}\left[\lambda_n^2\sqrt{\frac{\omega_{11}}{\omega_{21}}t}\right],$$

and Si is the integral sine.

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