ՀԱՅԱՍՏԱՆԻ ԳԻՏՈՒԹՅՈՒՆՆԵՐԻ ԱԶԳԱՅԻՆ ԱԿԱԴԵՄԻԱՅԻ ՏԵՂԵԿԱԳԻՐ
ИЗВЕСТИЯ НАЦИОНАЛЬНОЙ АКАДЕМИИ НАУК АРМЕНИИ

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AXISYMMETRIC STABILITY OF CIRCULAR RING PLATE SUBJECTED TO MECHANICAL AND THERMAL LOADS

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Осесимметричная устойчивость круглой кольцевой плиты, подверженной механическим и термическим нагрузкам

Ключевые слова: круглая кольцевая пластина, упругая устойчивость, предельная нагрузка, критическое значение, прогиб.

Исследована возможность потери осесимметричной устойчивости круглой кольцевой пластины, подвергнутого воздействию равномерно изменяющегося по радиусу кольца теплового поля. Внутренная кромка кольца зажата или подвержена воздействию внешних механических сил. На внешнем крае круглого кольца равномерно распределенние силы действуют как растягивающая или сжимающая нагрузки. С учетом сформированного плоского напряженного состояния в круговом кольце для всех случаев внутренних напряжений в зонах сжатия определены значения продольного смещения. В случае общей нагрузки кольца напряжения выражаются как функциями Бесселя первого и второго порядка, так и цилиндрической функцией мнимых аргументов с действительными индексами. В некоторых частных случаях нагружения колец, уравнения устойчивости пластины перестраиваются в уравнения типа Эйлера, а напряжения в пластине выражаются через элементарные функции.

Ավետիսյան Ա.Ս., Ալեքսանյան Ռ.Կ., Ալեքսանյան Դ.Ռ.

Մեխանիկական և ջերմային բեռնավորված շրջանային օղակաձև սալի առանցքասիմետրիկ կայունությունը

Հիմնաբառեր: շրջանային օղակաձև սալ, առաձգական կայունություն, սահմանային բեռ, կրիտիկական արժեք, սալի ճկվածք:

Հետազոտվում է, շրջանային օղակաձև սալի առանցքասիմետրիկ կայունության կորուստի հնարավորությունը, երբ սալի շառավղով ազդում է հավասարաչափ փոփոխվող ջերմային դաշտ: Սալի ներքին եզրը ամրակցված է, կամ ենթարկված է արտաքին մեխանիկական ուժերի ազդեցությանը, իսկ արտաքին եզրում ազդում են հավասարաչափ բաշխված սեղմող կամ ձգող ուժեր: Կլոր շրջանային սալում ձևավորված հարթ լարվածային վիճակից ներքին լարումների բոլոր դեպքերի համար, սեղման տիրույթներում որոշվում են շառավղային տեղափոխությունների արժեքները: Կլոր սալի ընդհանուր բեռնավորման դեպքում, լարումները արտահայտվում են ինչպես Բեսսելի առաջին և երկրորդ սեռի, այնպես էլ իրական ինդեքսներով, կեղծ փոփոխականի գլանային ֆունկցիաների միջոցով: Կլոր սալի մասնակի բեռնավորման որոշ դեպքերում, սալի կայունության հավասարումը ստանում է Էյլերի հավասարման տեսք, իսկ լարումները սալում արտահայտվում են էլեմենտար ֆունկցիաներով:

The possibility of loss of the axisymmetric stability of a circular ring plate is investigated, when uniformly varying thermal field impacts over plate radius. The inner border of the plate is clamped or is exposed to the impact of external mechanical forces. Uniformly distributed tensile or compressive forces impact at the outer border of the plate. Taking into account the formed flat stress state in the circular ring plate, the values of the longitudinal displacements are determined for all cases of internal stresses in the compression zones. In the case of the general loading of the ring plate, the stresses are expressed by both first-order and second-order Bessel's functions, as well as by cylindrical functions of imaginary arguments with real indices. In the cases of ring plate partial loading, the equations of plate stability get the look of Euler type equations, and the stresses in the plate are expressed by elementary functions.

Introduction

The problem of circular ring plate stability subjected to the compressive forces at the mid-plane first time was investigated in [1]. The problem of axisymmetric stability loss of the circular ring plate subjected to the uniformly distributed compressive forces is considered in [2]. The general case of the above-mentioned problem is in paper [3]. The investigations of the problems of axisymmetric and non-axisymmetric stability of ring plate have been done in [4, 5] for two different cases: a) the circular ring plate is subjected to the uniformly distributed compressive forces along the outer border of the plate; b) the circular ring plate is subjected to the uniformly distributed tensile (radial) forces along the inner border of the plate.

The generalized problem of stability of the circular ring plate subjected to the uniformly distributed tensile and compressive (radial) forces along the outer and inner borders is considered in [6]. In this article, the various cases of boundary conditions are satisfied taking into account the longitudinal displacements determined from the formulas of plate stability problem. Equations of critical values of unknown parameters are received from the conditions of non-zero solution of the homogeneous equations' system for integration constants.

In technical report [7], an analysis of the stability of circular ring plates under uniformly distributed radial load is presented. It was found that, in general, the critical mode shapes are combinations of in- and out-of-plane displacements and they occur when loads considerably are below the classical (in-plane) critical load. In paper [8], the stability of circular ring plate, pre-stressed by temperature-like intrinsic deformation, is studied using the equations of the nonlinear theory of rods. The temperature gradient in the radial direction results in a bending moment. The analytical solutions are successfully compared against results of finite element simulations for a shell model of the ring.

In papers [9,10] the linearized problems on the stability of a circular ring sandwich of symmetric structure under axially symmetric temperature field, inhomogeneous through the core thickness, are stated and their analytical solutions are given. The deformation processes for the load-carrying layers are described by the Kirchhoff-Love model. For the core of arbitrary thickness, the deformation process is described - by two models, namely by the equations of the plane problem of elasticity theory and by the model of a transversely soft layer of arbitrary thickness.

1. The problem statement. Plane stress state of circular ring plate.

Let us consider a circular ring plate with the thickness h, which is clamped at the inner border $r = b$, and is subjected to the tensile or compressive uniformly distributed load with intensity P at outer border $r = a$. The plate is also exposed to uniformly varying temperature along radius of the ring plate. The plane stress state of axisymmetric circular ring plate is characterized by the direction of radius via the non-zero component of displacement ***u*** at the middle plane of the plate [11].

$$u_r = C_1 r + \frac{C_2}{r}, \quad b \le r \le a \tag{1.1}$$

Corresponding to the arising internal stresses σ_r and σ_θ, the arising at the circular ring plate internal efforts $N_r(r)$, $N_\theta(r)$ and $N_{r\theta}(r)$ are determined by the following formulas

$$N_r(r) = \frac{Eh}{1-\nu^2}\left[\frac{du_r}{dr} + \nu\frac{u_r}{r} - (1+\nu)\alpha T\right]$$

$$N_\theta(r) = \frac{Eh}{1-\nu^2}\left[\frac{u_r}{r} + \nu\frac{du_r}{dr} - (1+\nu)\alpha T\right] \tag{1.2}$$

$$N_{r\theta}(r) \equiv 0\,.$$

Here, E is the modulus of elasticity, ν is the Poisson's ratio, α is the coefficient of thermal expansion, T is the temperature change correspondingly for a plate material. Based on (1.1) and (1.2), as well as satisfying the boundary conditions on the edge of the circular ring plate

$$u\big|_{r=b} = 0, \quad N_r\big|_{r=a} = p, \tag{1.3}$$

we obtain the following relations for the cutting forces:

$$N_r(r) = \frac{Eh}{(1+\nu)\delta}\left[P_0(1+\frac{b^2\nu^*}{r^2}) + \alpha T l^2(\frac{a^2}{r^2}-1)\right],$$

$$N_\theta(r) = \frac{Eh}{(1+\nu)\delta}\left[P_0(1-\frac{b^2\nu^*}{r^2}) - \alpha T l^2(\frac{a^2}{r^2}+1)\right] \quad . \tag{1.4}$$

Here, $\nu^* = (1-\nu)/(1+\nu)$, $\delta = 1+\nu^* l^2$, $P_0 = P(1+\nu)/Eh$, and $l = b/a$ are dimensionless parameters of the circular ring plate.

If either tensile or compressive forces ($P > 0$ or $P < 0$) impact on the outer border of the circular ring plate and either steady increase or steady decrease of temperature ($T > 0$ or $T < 0$) is influencing the thermal expansion of the circular ring plate, the forces defined by equation (1.4) on the plane of the circular ring plate can be tensile or compressive, i.e. the strips (zones) of tension or compression forces are located on the circular ring plate. For any force, the presence of compressive zone is pointing out the lost of stability of the circular ring plate. Therefore, if $P_0 > 0$ and $T > 0$ then $N_r > 0$ for $r \in [b;a]$, meanwhile N_θ for $[b;a]$ can have a compressive zone. Furthermore, if $P_0 < 0$ and $T < 0$ then $N_r < 0$ for $r \in [b;a]$, meanwhile N_θ can change the sign.

Similar to the above-mentioned consideration, it is easy to prove that for P_0 and T having opposite signs, N_r and N_θ in the interval $[b;a]$ can contain compressive zones. Consequently, in any case, the circular ring plate can lose the stability.

The forces defined by formula (1.4) can be expressed as follows:

$$N_{r;\theta}(r) = \frac{Eh}{(1+\nu)\delta}\left[(P_0 - \alpha T l^2) \pm (P_0 \nu^* + \alpha T)\frac{b^2}{r^2}\right] \tag{1.5}$$

2. The problem of stability lost of axisymmetric circular ring plate

In the problem of stability lost of the axisymmetric circular ring plate losing its stability, the deflection $\mathrm{w}(r)$ satisfies to the following equation [2-6]

$$D\Delta\Delta \mathrm{w}(r) = N_r \frac{d^2\mathrm{w}(r)}{dr^2} + N_\theta \frac{1}{r}\frac{d\mathrm{w}(r)}{dr}, \tag{2.1}$$

where $D = Eh^3/12(1-\nu^2)$ is cylindrical flexural rigidity of the circular ring plate, and $\Delta(\circ) = \left(d^2/dr^2\right) + \left(1/r\right)\cdot\left(d/dr\right)$ is a one-dimensional differential operator.

Based on expressions (1.5), the equation (2.1) can be expressed in the following way:

$$\Delta\Delta \mathrm{w}(r) = \frac{Eh(P_0 - \alpha T l^2)}{D(1+\nu)\delta}\left[\left(1 \pm \frac{\gamma^2 b^2}{r^2}\right)\frac{d^2\mathrm{w}(r)}{dr^2} + \left(1 \mp \frac{\gamma^2 b^2}{r^2}\right)\frac{1}{r}\frac{d\mathrm{w}(r)}{dr}\right] \tag{2.2}$$

where "+" sign should be chosen for the values $(P_0 - \alpha T l^2)$ and $(P_0\nu^* + \alpha T)$ having the same sign; "-" sign should be chosen for the values having opposite signs, and $\gamma^2 = \left|P_0\nu^* + \alpha T\right| / \left|P_0 - \alpha T l^2\right|$.

The following status options of equation (2.2) determining for $\mathrm{w}(r)$ are considered:

I. if $P_0 - \alpha T l^2 > 0$, and $P_0\nu^* + \alpha T > 0$ then

$$\Delta\Delta \mathrm{w}(r) - \beta^2\left[\left(1 + \frac{\gamma^2 b^2}{r^2}\right)\frac{d^2\mathrm{w}(r)}{dr^2} + \left(1 - \frac{\gamma^2 b^2}{r^2}\right)\frac{1}{r}\frac{d\mathrm{w}(r)}{dr}\right] = 0. \tag{2.3}$$

II. if $P_0 - \alpha T l^2 > 0$ and $P_0\nu^* + \alpha T < 0$ then

$$\Delta\Delta \mathrm{w}(r) - \beta^2\left[\left(1 - \frac{\gamma^2 b^2}{r^2}\right)\frac{d^2\mathrm{w}(r)}{dr^2} + \left(1 + \frac{\gamma^2 b^2}{r^2}\right)\frac{1}{r}\frac{d\mathrm{w}(r)}{dr}\right] = 0. \tag{2.4}$$

where $\beta^2 = 12(1-\nu)(P_0 - \alpha T l^2)/h^2\delta$.

III. if $P_0 - \alpha T l^2 < 0$ and $P_0\nu^* + \alpha T < 0$ then

$$\Delta\Delta \mathrm{w}(r) + \beta_1^2\left[\left(1 + \frac{\gamma^2 b^2}{r^2}\right)\frac{d^2\mathrm{w}(r)}{dr^2} + \left(1 - \frac{\gamma^2 b^2}{r^2}\right)\frac{1}{r}\frac{d\mathrm{w}(r)}{dr}\right] = 0. \tag{2.5}$$

IV. if $P_0-\alpha Tl^2<0$ and $P_0\nu^*+\alpha T>0$ then

$$\Delta\Delta \mathrm{w}(r)+\beta_1^2\left[\left(1-\frac{\gamma^2 b^2}{r^2}\right)\frac{d^2\mathrm{w}(r)}{dr^2}+\left(1+\frac{\gamma^2 b^2}{r^2}\right)\frac{1}{r}\frac{d\mathrm{w}(r)}{dr}\right]=0, \qquad (2.6)$$

where $\beta_1^2=12(1-\nu)\left|P_0-\alpha Tl^2\right|/h^2\delta$.

It is evident that here the following limiting cases are possible.

1. In case of $P_0-\alpha Tl^2=0$ and $P_0\nu^*+\alpha T\neq 0$, the equation (2.1) can be expressed as follows

$$\Delta\Delta \mathrm{w}(r)-H^2\left(\frac{1}{r^2}\frac{d^2\mathrm{w}(r)}{dr^2}-\frac{1}{r^2}\frac{d\mathrm{w}(r)}{dr}\right)=0, \text{ for } P_0\nu^*+\alpha T>0, \qquad (2.7)$$

where $H^2=12(1-\nu)b^2(P_0\nu^*+\alpha T)/h^2\delta$, and as follows

$$\Delta\Delta \mathrm{w}(r)+\bar{H}^2\left(\frac{1}{r^2}\frac{d^2\mathrm{w}(r)}{dr^2}-\frac{1}{r}\frac{d\mathrm{w}(r)}{dr}\right)=0, \text{ for } P_0\nu^*+\alpha T<0, \qquad (2.8)$$

where $\bar{H}^2=12(1-\nu)b^2\left|P_0\nu^*+\alpha T\right|/h^2\delta$.

2. In the case of $P_0-\alpha Tl^2\neq 0$ and $P_0\nu^*+\alpha T=0$: if $P_0-\alpha Tl^2>0$ then $N_r=N_\theta=Eh\left(P_0-\alpha Tl^2\right)/\delta\left(1+\nu\right)>0$, in $r\in[a,b]$. Therefore, the plate is subjected to all-around tension; meanwhile, the plate doesn't lose the stability.

3. If $P_0-\alpha Tl^2<0$ and $P_0\nu^*+\alpha T=0$ then $N_r=N_\theta=Eh\left(P_0-\alpha Tl^2\right)/\left(1+\nu\right)\delta<0$. In this case, the circular ring plate is subjected to all-around compression. Consequently, the circular ring plate can lose the stability [12]. For this purpose, the equation (2.1) should be expressed as follows

$$\Delta\Delta \mathrm{w}+H_1^2\Delta \mathrm{w}=0, \text{ where } H_1^2=12(1-\nu)\left|P_0-\alpha Tl^2\right|/(1+\nu)\delta \qquad (2.9)$$

4. If $P_0-\alpha Tl^2=0$ and $P_0\nu^*+\alpha T=0$ then $P_0=0$ and $\alpha T=0$.

In this case, the circular ring plate is free from external mechanical and thermal actions and it will by stabile.

3. Integration of stability equations of the circular ring plate.

Replacing variables $x=\beta\cdot r$, equations (2.3) and (2.4) can be expressed as follows:

$$\Delta_x\Delta_x \mathrm{w}(x)-\left[\left(1+\frac{d^2\beta^2}{x^2}\right)\frac{d^2\mathrm{w}(x)}{dx^2}+\left(1-\frac{d^2\beta^2}{x^2}\right)\frac{1}{x}\frac{d\mathrm{w}(x)}{dx}\right]=0, \qquad (3.1)$$

$$\Delta_x\Delta_x \mathrm{w}(x)-\left[\left(1-\frac{d^2\beta^2}{x^2}\right)\frac{d^2\mathrm{w}(x)}{dx^2}+\left(1+\frac{d^2\beta^2}{x^2}\right)\frac{1}{x}\frac{d\mathrm{w}(x)}{dx}\right]=0\,, \tag{3.2}$$

where $d^2=\gamma^2 b^2$, $\Delta_x=\left(d^2/dx^2\right)+\left(1/x\right)\left(d/dx\right)$ - is a one-dimensional differential operator. If denote $z(x)=d\mathrm{w}(x)/dx$, the equations (3.1) and (3.2) can be represented as follows:

$$x\left[x^2z''+xz'-(\eta_1^2+x^2)z\right]'-\left[x^2z''+xz'-(\eta_1^2+x^2)z\right]=0\,, \tag{3.3}$$

$$x\left[x^2z''+xz'-(\eta_2^2+x^2)z\right]'-\left[x^2z''+xz'-(\eta_2^2+x^2)z\right]=0\,, \tag{3.4}$$

where $\eta_{1,2}^2=1\pm d^2\beta^2$.

Integrating equations (3.3) and (3.4), the following equations can be obtained:

$$x^2z''+xz'-(\eta_1^2+x^2)z=Cx\,, \tag{3.5}$$

$$x^2z''+xz'-(\eta_2^2+x^2)z=\overline{C}x\,, \tag{3.6}$$

here C and $\overline{C}$ are integration constants. The solution of equations (3.5) and (3.6) should be determined by the following expressions.

$$\begin{gathered} z=C_1(x)I_{\eta_1}(x)+C_2(x)K_{\eta_1}(x) \\ z=E_1(x)I_{\eta_2}(x)+E_2(x)K_{\eta_2}(x)\,, \end{gathered} \tag{3.7}$$

where $I_\eta(x)$ and $K_\eta(x)$ are the cylindrical function of imaginary arguments with the real indexes; $C_i(x),\ E_i(x)\ (i=1,2)$ are arbitrary functions defined by the variation method of arbitrary constants.

The solutions of equations (3.5) and (3.6) can be expressed as follows:

$$\begin{gathered} z=\overline{C_1}I_{\eta_1}(x)+\overline{C_2}K_{\eta_1}(x)+Cf_1(x) \\ z=\overline{E_1}I_{\eta_2}(x)+\overline{E_2}K_{\eta_2}(x)+\overline{C}f_2(x)\,. \end{gathered} \tag{3.8}$$

The corresponding deflections can be defined by the following equations:

$$\begin{gathered} \mathrm{w}(x)=\int[\overline{C_1}I_{\eta_1}(x)+\overline{C_2}K_{\eta_1}(x)]dx+C\int f_1(x)dx+H_1 \\ \mathrm{w}(x)=\int[\overline{E_1}I_{\eta_2}(x)+\overline{E_2}K_{\eta_2}(x)]dx+\overline{C}\int f_2(x)dx+H_2\,, \end{gathered} \tag{3.9}$$

where $\overline{C}_i;\overline{E}_i;\overline{H}_i(i=1,2)$ are integration constants, and

$$f_i(x)=I_{\eta_i}(x)\int K_{\eta_i}(x)dx-K_{\eta_i}(x)\int I_{\eta_i}(x)dx \quad (i=1,2)\,.$$

Denoting $x_1=\beta_1 r$, equations (2.5) and (2.6) can be represented as follows:

$$\Delta_x\Delta_x \mathrm{w}(x_1)+\left(1+\frac{d^2\beta_1^2}{x_1^2}\right)\frac{d^2\mathrm{w}(x_1)}{dx_1^2}+\left(1-\frac{d^2\beta_1^2}{x_1^2}\right)\frac{1}{x_1}\frac{d\mathrm{w}(x_1)}{dx_1}=0, \qquad (3.10)$$

$$\Delta_x\Delta_x \mathrm{w}(x_1)+\left(1-\frac{d^2\beta_1^2}{x_1^2}\right)\frac{d^2\mathrm{w}(x_1)}{dx_1^2}+\left(1+\frac{d^2\beta_1^2}{x_1^2}\right)\frac{1}{x_1}\frac{d\mathrm{w}(x_1)}{dx_1}=0. \qquad (3.11)$$

The integration of equations (3.10) and (3.11) can be reduced to the integration of Bessel's non-homogeneous equations.

$$x_1^2 z''+x_1 z'-(\xi_1^2-x_1^2)z=C_0x_1,$$
$$x_1^2 z''+x_1 z'-(\xi_2^2-x_1^2)z=\bar{C}_0x_1, \qquad (3.12)$$

where $z(x_1)=d\mathrm{w}(x_1)/dx_1$, C_0, $\bar{C}_0$ are integration constants.

The generalized solutions of equations (3.12) are the following

$$z(x_1)=\bar{E}_1J_{\xi_1}(x_1)+\bar{E}_2Y_{\xi_1}(x_1)+\left(\pi C_0/2\right)\phi_1(x_1)$$
$$z(x_1)=\bar{H}_1J_{\xi_2}(x_1)+\bar{H}_2Y_{\xi_2}(x_1)+\left(\pi\bar{C}_0/2\right)\phi_2(x_1) \qquad (3.13)$$

The corresponding deflections can be obtained from the following relations

$$\mathrm{w}(x_1)=\int\left[\overline{E_1}J_{\xi_1}(x_1)+\overline{E_2}Y_{\xi_1}(x_1)\right]dx_1+\frac{\pi C_0}{2}\int\phi_1(x_1)dx_1+E_0$$
$$\mathrm{w}(x_1)=\int\left[\overline{H_1}J_{\xi_2}(x_1)+\overline{H_2}Y_{\xi_2}(x_1)\right]dx_1+\frac{\pi\overline{C_0}}{2}\int\phi_2(x_1)dx_1+H_0, \qquad (3.14)$$

where $J_\xi(x_1)$ and $Y_\xi(x)$ are the first-order and the second-order Bessel's functions; $\bar{E}_i, \bar{H}_i, E_0, H_0$ $(i=1,2)$ are integration constants; and

$\phi_i(x_1)=Y_{\xi i}(x_1)\int J_{\xi i}(x_1)dx_1$ - $J_{\xi i}(x_1)\int Y_{\xi i}(x_1)dx_1$ $(i=1,2)$.

For the first limiting case ($P_0-\alpha Tl^2=0$, $P_0\nu^*+\alpha T\neq 0$) and $P_0\nu^*+\alpha T>0$, the force $\mathrm{w}(x_1)$ is obeying equation (2.7). Meanwhile, for $P_0\nu^*+\alpha T<0$, the force $\mathrm{w}(x_1)$ is obeying equation (2.8). The equation (2.7) is an equation of Eulerian kind:

$$\mathrm{w}^{IV}+\frac{2}{r}\mathrm{w}^{III}-(1+H^2)\frac{1}{r^2}\mathrm{w}^{II}+(1+H^2)\frac{1}{r^3}\mathrm{w}^{I}=0, \qquad (3.15)$$

where $H^2=12(1-\nu)b^2(P_0\nu^*+\alpha T)/h^2(1+\nu^*l^2)$.

Denoting $x=r/a$, $l\le x\le 1$, $l=b/a$, equation (3.15) can be represented as follows:

$$x^3\mathrm{w}^{IV}+2x^2\mathrm{w}^{III}-x(1+H^2)\mathrm{w}^{II}+(1+H^2)\mathrm{w}^{I}=0. \qquad (3.16)$$

The generalized solution of eq. (3.16) is the following

$$w(x)=C_1+C_2x^2+C_3x^{1+k}+C_4x^{1-k}, \quad (3.17)$$

where $k=\sqrt{1+H^2}$.

In case of $P_0\nu^*+\alpha T<0$, the equation (2.8) can be represented as follows

$$x^3w^{IV}(x)+2x^2w^{III}(x)-x(1-\bar{H}^2)w^{II}+(1-\bar{H}^2)w^{I}(x)=0, \quad (3.18)$$

where $\bar{H}^2=12(1-\nu)b^2\left|P_0\nu^*+\alpha T\right|/h^2(1+\nu^*l^2)$

In the case of $1-\bar{H}^2>0$, the deflection can be written using the generalized solution of equation (3.18)

$$w(x)=B_1+B_2x^2+B_3x^{1+k}+B_4x^{1-k}. \quad (3.19)$$

Similarly, in the case of $1-\bar{H}^2<0$, the corresponding solution of equation (3.18) can be obtained as

$$w(x)=D_1+D_2x^2+x\cdot[D_3\cos(\beta\ln x)+D_4\sin(\beta\ln x)], \quad (3.20)$$

where $\beta^2=\bar{H}^2-1$.

For $1-\bar{H}^2=0$, deflections can be defined as

$$w(x)=C_1x\cdot\ln x+C_2x^2+C_3x+C_4. \quad (3.21)$$

For the second limiting case $P_0-\alpha Tl^2\neq 0$ and $P_0\nu^*+\alpha T=0$, the circular ring plate can lose its stability, if $P_0-\alpha Tl^2<0$. For this case, the deflection can be determined by equation (2.9). Replacing variables $\rho=r/a$, $l\leq\rho\leq 1$, equation (2.9) can be represented as follows [12]:

$$\Delta_\rho\Delta_\rho w(x)+k^2\Delta_\rho w(x)=0, \quad (3.22)$$

Where $k^2=12a^2(1-\nu)\left|P_0-\alpha Tl^2\right|/h^2\left(1+\nu^*l^2\right)$,

$\Delta_\rho=d^2/d\rho^2+(1/\rho)(d/d\rho)$,

Denoting $x=k\rho$, the following equation is obtained:

$$\Delta_x\Delta_x w(x)+\Delta_x w(x)=0. \quad (3.23)$$

Meanwhile, denoting $w'=z$, the following equation is obtained:

$$x\left[x^2z''(x)+xz'(x)-(1-x^2)z(x)\right]'-\left[x^2z''(x)+xz'(x)-(1-x^2)z(x)\right]=0.$$

Integrating the last equation, the following equation is obtained:

$$x^2z''(x)+xz'(x)-(1-x^2)z(x)=C_0x, \quad (3.24)$$

where C_0 is an integration constant.

The generalized solution of equation (3.24) can be defined in the following way:

$$z(x)=A_1J_1(x)+B_1Y_1(x)+C_0/x. \quad (3.25)$$

Taking into account $z(x) = d\mathrm{w}(x)/dx$ and integrating eq. (3.25), the following equation can be obtained:

$$\mathrm{w}(x) = A_0 J_0(x) + B_0 Y_0(x) + C_0 \ln x + D_0, \quad (3.26)$$

where A_0, B_0, C_0, and D_0 are integration constants.

4. Determination of parameters critical values

Longitudinal displacements defined by the integration of stability equations of the circular ring plate should satisfy the boundary conditions on the plate borders.

In practice, the following cases of boundary conditions are considered (Figure 1. ÷ Figure 4.)

1.

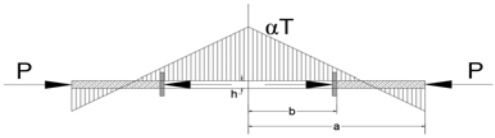

Figure 1. The internal border of the plate $r = b$ is clamped, and the external border of the plate $r = a$ is free

with the following boundary conditions,

$$\mathrm{w}\big|_{r=b} = 0, \quad \frac{d\mathrm{w}}{dr}\Big|_{r=b} = 0, \quad M_r\big|_{r=a} = 0, \quad Q_r\big|_{r=a} = 0 \quad (4.1)$$

2.

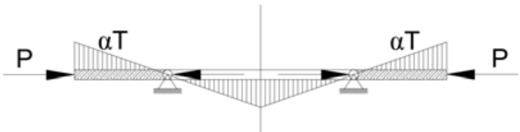

Figure 2. The internal border of the plate $r = b$ is hinged, and the external border of the plate $r = a$ is free.

with the following boundary conditions,

$$\mathrm{w}\big|_{r=b} = 0, \quad M_r\big|_{r=b} = 0, \quad M_r\big|_{r=a} = 0, \quad Q_r\big|_{r=a} = 0 \quad (4.2)$$

3.

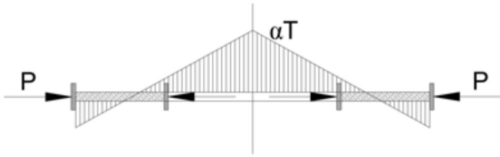

Figure 3. The borders of the plate are clamped.

with the following boundary conditions,

$$\mathrm{w}\big|_{r=b}=0\,,\quad \frac{d\mathrm{w}}{dr}\Big|_{r=b}=0\,,\quad \mathrm{w}\big|_{r=a}=0\,,\quad \frac{d\mathrm{w}}{dr}\Big|_{r=a}=0 \tag{4.3}$$

4.

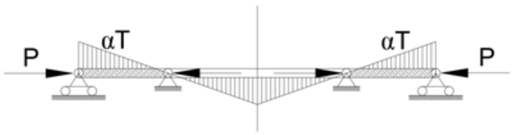

Figure 4. The borders of the plate are hinged

with the following boundary conditions,

$$\mathrm{w}\big|_{r=b}=0\,,\quad M_r\big|_{r=b}=0\,,\quad M_r\big|_{r=a}=0\,,\quad \mathrm{w}\big|_{r=a}=0 \tag{4.4}$$

In the problem of axisymmetric stability of the plate, bending moments and cutting forces through forces $\mathrm{w}(r)$ can be defined by the following equations:

$$M_r=-D\left(\frac{d^2\mathrm{w}}{dr^2}+\frac{\nu}{r}\frac{d\mathrm{w}}{dr}\right)\,,\qquad Q_r=-D\frac{d}{dr}\left(\Delta\mathrm{w}\right)\quad\text{, or}$$

$$M_r=-D\beta^2\left(\frac{d^2\mathrm{w}}{dx^2}+\frac{\nu}{x}\frac{d\mathrm{w}}{dx}\right)=-\frac{D\beta^2}{x}\left(xz'+\nu z\right); \tag{4.5}$$

$$Q_r=-D\beta^2\frac{d}{dx}\left(\frac{d^2\mathrm{w}}{dx^2}+\frac{1}{x}\frac{d\mathrm{w}}{dx}\right)=-\frac{D\beta^2}{x}\left(x^2z''+xz'-z\right);\ x=\beta r\frac{\delta y}{\delta x}.$$

In the view of the determination of critical values of parameters, let's consider the fourth option ($P_0-\alpha Tl^2<0$, $P_0\nu^*+\alpha T>0$). In this case, from equations (3.13) and (3.14), $z=\mathrm{w}'(x_1)$ and $\mathrm{w}(x_1)$ can be expressed as follows:

$$z(x_1) = \overline{H}_1 J_{\xi_2}(x_1) + \overline{H}_2 Y_{\xi_2}(x_1) + \frac{\pi \overline{C}_0}{2} \phi_2(x_1)\,; \tag{4.6}$$

$$\mathrm{w}(x_1) = \int_{\beta_1 b}^{x_1} \left[\overline{H}_1 J_{\xi_2}(x_1) + \overline{H}_2 Y_{\xi_2}(x_1) + \frac{\pi \overline{C}_2}{2} \phi_2(x_1) \right] dx_1 + \overline{H}_0\,, \tag{4.7}$$

Where $x_1 = \beta_1 r, \quad \xi_2 = \beta^2{}_1 d^2, \quad d^2 = \gamma^2 b^2, \quad \beta_1 d \le x_1 \le \beta_1 a$, and

$$\phi_2(x_1) = -J_{\xi_2}(x_1) \int_{\beta_1 b}^{x_1} Y_{\xi_2}(x)dx + Y_{\xi_2}(x_1) \int_{\beta_1 b}^{x_1} J_{\xi_2}(x)dx$$

Satisfying the boundary conditions (4.1) and based on equations (4.6) - (4.7), we can receive the following system of equations:

$$\begin{cases} H_0 = 0 \\ \overline{H}_1 J_{\xi_2}(\beta_1 b) + \overline{H}_2 Y_{\xi_2}(\beta_1 b) = 0 \\ \overline{H}_1 J_{\xi_2}(\beta_1 a) + \overline{H}_2 Y_{\xi_2}(\beta_1 a) + \\ + \dfrac{\pi \overline{C}_0}{2} \left[-J_{\xi_2}(\beta_1 a) \displaystyle\int_{\beta_1 b}^{\beta_1 a} Y_{\xi_2}(x)dx + Y_{\xi_2}(\beta_1 a) \int_{\beta_1 b}^{\beta_1 a} J_{\xi_2}(x)dx \right] - \dfrac{\overline{C}_0}{\beta_1 a(1-\gamma^2 l^2)} = 0 \\ \overline{H}_1 J'_{\xi_2}(\beta_1 a) + \overline{H}_2 Y'_{\xi_2}(\beta_1 a) + \\ + \dfrac{\pi \overline{C}_0}{2} \left[-J'_{\xi_2}(\beta_1 a) \displaystyle\int_{\beta_1 b}^{\beta_1 a} Y_{\xi_2}(x)dx + Y'_{\xi_2}(\beta_1 a) \int_{\beta_1 b}^{\beta_1 a} J_{\xi_2}(x)dx \right] + \dfrac{\nu \overline{C}_0}{\beta^2{}_1 a^2 (1-\gamma^2 l^2)} = 0 \end{cases} \tag{4.8}$$

For the boundary condition (4.2), the following system of equations can be obtained:

$$\begin{cases} \overline{H}_0 = 0 \\ \left[J'_{\xi_2}(\beta_1 b) + \nu J_{\xi_2}(\beta_1 b) \right] \overline{H}_1 + \left[Y'_{\xi_2}(\beta_1 b) + \nu Y_{\xi_2}(\beta_1 b) \right] \overline{H}_2 = 0 \\ \overline{H}_1 J'_{\xi_2}(\beta_1 a) + \overline{H}_2 Y'_{\xi_2}(\beta_1 a) + \\ + \dfrac{\pi \overline{C}_0}{2} \left[-J'_{\xi_2}(\beta_1 a) \displaystyle\int_{\beta_1 b}^{\beta_1 a} Y_{\xi_2}(x)dx + Y'_{\xi_2}(\beta_1 a) \int_{\beta_1 b}^{\beta_1 a} J_{\xi_2}(x)dx \right] + \dfrac{\nu \overline{C}_0}{\beta^2{}_1 a^2 (1-\gamma^2 l^2)} = 0 \\ \overline{H}_1 J_{\xi_2}(\beta_1 a) + \overline{H}_2 Y_{\xi_2}(\beta_1 a) + \\ + \dfrac{\pi \overline{C}_0}{2} \left[-J_{\xi_2}(\beta_1 a) \displaystyle\int_{\beta_1 b}^{\beta_1 a} Y_{\xi_2}(x)dx + Y_{\xi_2}(\beta_1 a) \int_{\beta_1 b}^{\beta_1 a} J_{\xi_2}(x)dx \right] - \dfrac{\overline{C}_0}{\beta_1 a(1-\gamma^2 l^2)} = 0 \end{cases} \tag{4.9}$$

Based on the existence condition of non-zero solution of the system of homogeneous equations for the integration constants obtained by equations (4.8) and (4.9), the transcendent equation for any random parameter can be determined taking into account that the rest of parameters have received specified values.

For the first limiting case ($P_0 - \alpha T l^2 = 0$, $P_0 \nu^* + \alpha T \neq 0$), the force $\mathrm{w}(x)$ can be obtained by equations (3.17) and (3.19) in case of $P_0 \nu^* - \alpha T l^2 < 0$. Determining the force $\mathrm{w}(x)$ by equation (3.19) and satisfying the boundary conditions (4.3), for the determination of integration parameters B_i the following system of homogeneous equations will be received.

$$\begin{cases} B_1 + B_2 l^2 + B_3 l^{1+\bar{k}} + B_4 l^{1-\bar{k}} = 0 \\ 2lB_2 + B_3(1+\bar{k})l^{\bar{k}} + B_4(1-\bar{k})l^{-k} = 0 \\ B_1 + B_2 + B_3 + B_4 = 0 \\ 2B_2 + (1+\bar{k})B_3 + (1-\bar{k})B_4 = 0 \end{cases} \tag{4.10}$$

Taking into account that

$$\begin{cases} P_0 - \alpha T l^2 = 0 \\ P_0 \nu^* + \alpha T < 0 \end{cases} \quad \begin{cases} P_0 = \alpha T l^2 \\ \alpha T (1 + \nu^* l^2) > 0 \end{cases} \quad \Rightarrow \quad \begin{cases} P_0 > 0 \\ T < 0 \end{cases},$$

and based on the existence condition of non-zero solution of the system of homogeneous equations (4.10), the following equation can be obtained:

$$-4\bar{k}l + (1+\bar{k}^2)(1-l^2)\mathrm{sh}(\bar{k}\ln l) + 2\bar{k}(1+l^2)\mathrm{ch}(\bar{k}\ln l) = 0 . \tag{4.11}$$

Consequently, for the plate losing the plane stability, either P_0 or the minimal critical value $|T|$ can be determined.

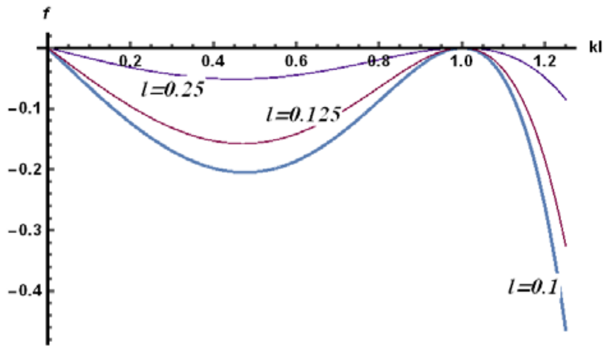

Figure 5. Characteristic stability curves of a circular ring plate with a joint thermo mechanical load from equation (4.11) and satisfying the boundary conditions (4.3)

Replacing B_i with C_i and $\bar{k}$ with k in equations (4.10) and (4.11), C_i constants and the critical values of parameters can be determined if the force is stated by equation (3.17).

The characteristic lines in accordance with equation (4.11), for different linear dimensions $l = b/a$ of a circular disc, are shown in Figure 5.

Taking into account that

$$\begin{cases} P_0 - \alpha T l^2 < 0 \\ P_0 \nu^* + \alpha T = 0 \end{cases} \Rightarrow \begin{cases} P_0 (1 + \nu^* l^2) < 0 \\ \alpha T = -P_0 \nu^* \end{cases} \Rightarrow \begin{cases} P_0 < 0 \\ \alpha T > 0 \end{cases},$$ and based on the second limiting case of the plate, the deflection $\mathrm{w}(x)$ and $z = \mathrm{w}'(x)$ obtained from the equations (3.25) - (3.26), as well as, satisfying the boundary condition (4.1), the following system of equations can be defined:

$$A_0 J_0(kl) + B_0 Y_0(kl) + C_0 \ln(kl) + D_0 = 0$$

$$A_0 J_1(kl) + B_0 Y_1(kl) - \frac{C_0}{kl} = 0$$

$$A_0 J_1(k) + B_0 Y_1(k) = 0$$

$$A_0 J_1'(k) + B_0 Y_1'(k) - C_0 \frac{1-\nu}{k^2} = 0, \tag{4.12}$$

where $k^2 = 12(1-\nu)\left|P_0\right| a^2 / h^2 = 12(1-\nu)\alpha T a^2 / h^2 \nu^*$ (4.13)

Based on the existence condition of non-zero solution of the system of homogeneous equations (4.12), the following transcendent equation can be obtained:

$$J_1(kl)Y_1(k) - J_1(k)Y_1(kl) = \frac{2}{\pi l(1-\nu)}. \tag{4.14}$$

Based on the known physical mechanical constants and geometrical parameters, the minimal positive value of k can be gotten from the equation (4.14), and either the value of the force $\left|P_0\right|_{cr}$ or the value of temperature T_{cr} corresponding to the value of k can be obtained from equation (4.13).

The characteristic lines in accordance with equation (4.14), for different linear dimensions $l = b/a$ of a circular disc, are shown in Figure 6.

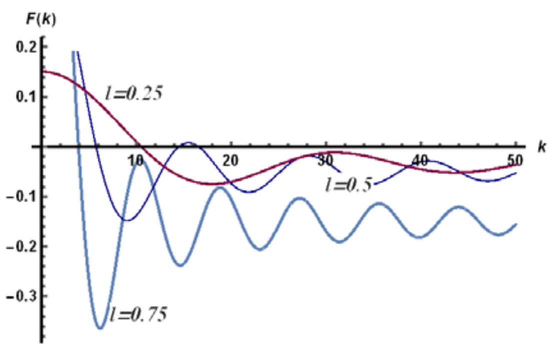

Figure 6. Characteristic stability curves of a circular ring plate with a joint thermo mechanical load from equation (4.14) and satisfying the boundary conditions (4.1)

Therefore, if the inner radius of circular ring plate $b \to 0$, i.e. the circular ring plate is reduced to the circular plate, the components of displacement $u(r)$ and internal forces corresponding to $u(r)$ will act on the mid-plane of the plate.

Then, we will have the following values:

$$u = C_1 r \ , \ N_r = N_\theta = \frac{Eh}{1-\nu}(C_1 - \alpha T) \ , \ N_{r\theta} = 0 \ , \ 0 \le r \le a$$

Satisfying the boundary condition $N_r\big|_{r=a} = \underline{P}$, the following expressions are obtained:

$$C_1 = P(1-\nu)/Eh + \alpha T \ , \ N_r = N_\theta = P \quad 0 \le r \le a \, .$$

Therefore, the internal forces are not dependent on the changes of temperature (in the case of uniform thermal variations, stresses do not emerge in the body). The circular plate can lose its stability if $P < 0$, so that the plate is subjected to the uniformly distributed compression $N_r = N_\theta = P < 0$.

The problem of axisymmetric stability loss of the circular plate clamped on the border subjected to the uniformly distributed forces in the direction of the diameter edge is considered by Bryan [13]. The problem of plate clamped on the border is investigated by Dinnik [14].

Conclusion.

The values of the longitudinal displacement are determined based in the formed flat stress state in a circular ring plate for all cases of internal stresses in the compression zones.

In the case of a general load, the stress rings are expressed by both first-order and second-order Bessel functions, and by cylindrical functions of imaginary arguments with real indices. In the cases of partial loading of the ring plates, the stability equations of the plate take the look of Euler type equations, and the stresses in the plate are expressed by elementary functions.

The internal forces are not dependent on the changes of temperature (in the case of uniform thermal variations, stresses don't emerge in the body free of the external constrains). The circular plate can lose its stability if $P < 0$, so that the plate is subjected to the uniformly distributed compression $N_r = N_\theta = P < 0$.

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