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ИЗВЕСТИЯ НАЦИОНАЛЬНОЙ АКАДЕМИИ НАУК АРМЕНИИ

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ON 3D THEORY OF ACOUSTIC METAMATERIALS WITH A TRIPLE-PERIODIC SYSTEM OF INTERIOR OBSTACLES

Remizov M.Yu., Sumbatyan M.A.

Keywords: doubly periodic array of cracks; one-mode propagation; integral equation; semi-analytical method; reflection and transmission coefficients; acoustic filters
Ключевые слова: двояко-периодический массив трещин; низкочастотный режим; интегральное уравнение; полуаналитический метод; коэффициент отражения и прохождения; акустический фильтр
Բանալի բառեր՝ ճաքերի երկպարբերական զանգված; ցածր հաճախությունների ռեժիմ; ինտեգրալ հավասարում; կիսավերլուծական եղանակ; արտացոլման և անցման գործակից; ձայնային ֆիլտր

Ремизов М.Ю., Сумбатян М.А.
К теории акустических метаматериалов с трояко-периодической системой внутренних неоднородностей
В трёхмерной постановке предлагается аналитический метод исследования распространения плоской упругой волны через систему произвольного конечного числа параллельных двоякопериодических идентичных массивов трещин. В условиях низкочастотного режима задача сводится к системе интегральных уравнений на одной выделенной типовой прямоугольной трещине. Полуаналитический метод, разработанный ранее для трёхмерных скалярных и плоских упругих задач, приводит к явным аналитическим представлениям для волнового поля и параметров рассеяния - коэффициентов отражения и прохождения.

Ռեմիզով Մ.Յու., Սումբատյան Մ.Ա.
Ներքին անհամասեռությունների եռապարբերական համակարգով ակուստիկ մետանյութերի տեսության վերաբերյալ
Եռաչափ դրվածքով առաջարկվում է ճաքերի կամայական վերջավոր թվով զուգահեռ երկպարբերական նույնական զանգվածներով հարթ առաձգական ալիքի տարածման հետազոտման վերլուծական եղանակ: Ցածր հաճախականության ռեժիմի պայմաններում խնդիրը բերվում է մեկ առանձնացված տիպային ուղղանկյունային ճաքի վրա ինտեգրալ հավասարումների համակարգի: Եռաչափ սկալյար և առաձգական հարթ խնդիրների համար նախկինում մշակված կիսավերլուծական եղանակը բերում է բացահայտ անալիտիկ արտահայտությունների ալիքային դաշտի և ցրման պարամետրերի՝ արտացոլման և անցման գործակիցների, համար:

In a three-dimensional (3-D) context an analytical approach is proposed to study the propagation of elastic plane wave through a system of any finite number of parallel doubly-periodic identical gratings of coplanar cracks. In the low frequency range the problem is reduced to a system of integral equations holding over the crack of a chosen elementary rectangular cell of the grating. The semi-analytical method previously introduced for scalar and elastic 2-D problems gives an explicit representations for the wave field and the scattering parameters - the reflection and transmission coefficients.

Introduction

The investigation of the scattering phenomena for waves propagating through media containing gratings of periodic geometry is nowadays a subject of great interest in many fields of engineering science, particularly - with impressive mechanical, electromagnetic and acoustical applications. Different numerical methods have been applied in the 2-D problems for periodic apertures [1-5]. Introducing various approximations, valid in the one-mode regime, the authors obtain some analytical solutions and develop respective formulas for the reflection and transmission coefficients in an explicit form.

The papers [6-9] provide explicit analytical formulas for reflection and transmission coefficients in the one-mode case for scalar acoustic or electromagnetic waves penetrating through doubly and triple-periodic arrays of arbitrary-shaped apertures and volumetric obstacles. In the two-dimensional in-plane problems on wave propagation through a periodic array of screens in elastic solids the works [10,11] can be cited for a single-periodic system of cracks and the works [12,13] -- for the doubly-periodic geometry.

The present work continues to study the 3-D elastic case for several doubly periodic arrays of coplanar cracks of arbitrary configuration, parallel to each other. The discussion is restricted to rectangular cracks and any finite number of parallel identical arrays. The topic of the problem is connected with [14,15] and some other published papers, but the mathematical technique is different. The wave process is harmonic in time, and all physical quantities contain the factor $e^{-i\omega t}$, which is further omitted, for the sake of brevity. Like in some previous works, (a) only one-mode propagation with normal incidence is considered ($ak_2, ck_2 < \pi$), where k_2 is the wave number of the transverse wave, $2a;2c$ are the periods of the grating; (b) the vertical cracked planes are sufficiently distant from each other, so that the ratios $D/a, D/c$ are comparatively large, where D is the distance between the neighbor arrays. In frames of the proposed approach, the problem is reduced to a dual hyper-singular integral equation, whose semi-analytical solution permits an explicit-form representation for the reflection and transmission coefficients.

The problem under consideration is connected with the theory and the practice of the so-called «acoustic metamaterials», which may possess a property of acoustic filter with a cutoff of the propagating wave over certain frequency intervals, due to their specific internal structure. This phenomenon for the elastic triple-periodic structures was recently discovered experimentally, being presented in [16]. Some fundamental aspects of the acoustic metamaterials are discussed, among many other publications, in [17-19].

Formulation of the problem and reducing to an integral equation

Let us consider a 3-D medium, which consists of M infinite planes, located at $x = 0, D, 2D, \ldots (M-1)D$, each containing a two-dimensional infinite periodic array of co-planar cracks, symmetric with respect to axes y and z. The distance between the systems of cracks, forming the third period is D. The period of the grating along axis y is $2a$, and along axis z is $2c$. If we study the incidence of the longitudinal plane wave $e^{i(k_1 x - \omega t)}$ upon the positive direction of axis x, then the problem is obviously equivalent (due to a symmetry) to a single waveguide of width $2a$ along axis y and $2c$ along axis z, (see Fig.1). Let us assume that a longitudinal plane wave of the form

$$\phi_0 = e^{ik_1 x}, \quad \psi_l = 0, \ (l = 1,2,3), \quad \Delta\phi + k_1^2\phi = 0, \quad \Delta\psi + k_2^2\psi = 0, \qquad (1)$$

is entering from $-\infty$, generating the scattered fields in front of the first array $(x < 0)$, inside the structure $((s-1)D < x < sD;\ s = 1,...M-1)$ and behind the last one $(x > (M-1)D)$. Then the Lame potentials, satisfying the Helmholtz equations in the respective domain, can be represented as the Fourier trigonometric series expansions along y and z variables:

$$x<0:\quad \varphi^l = e^{ik_1x} + \mathrm{R}e^{-ik_1x} + \sum_{n+j>0} A_{nj} e^{q_{nj}x} \cos(a_n y)\cos(c_j z),$$

$$\psi_1^l = \sum_{n+j>0} B_{nj}^1 e^{r_{nj}x} \sin(a_n y)\sin(c_j z),$$

$$\psi_2^l = \sum_{n+j>0} B_{nj}^2 e^{r_{nj}x} \cos(a_n y)\sin(c_j z), \qquad (2a)$$

$$\psi_3^l = \sum_{n+j>0} B_{nj}^3 e^{r_{nj}x} \sin(a_n y)\cos(c_j z),$$

$$(s-1)D < x < sD;\quad s = 1,...M-1:$$

$$\varphi^s = e^{ik_1x} + F_0^s \cos k_1[x-(s-1)D] + H_0^s \cos k_1(x-sD) +$$

$$+\sum_{n+j>0}^{\infty} \left\{F_{nj}^s \mathrm{ch}\, q_{nj}[x-(s-1)D] + H_{nj}^s \mathrm{ch}\, q_{nj}(x-sD)\right\} \cos(a_n y)\cos(c_j z),$$

$$\psi_1^s = \sum_{n+j>0}^{\infty} \left\{G_{nj}^s \mathrm{ch}\, r_{nj}[x-(s-1)D] + P_{nj}^s \mathrm{ch}\, r_{nj}(x-sD)\right\} \sin(a_n y)\sin(c_j z), \quad (2b)$$

$$\psi_2^s = \sum_{n+j>0}^{\infty} \left\{V_{nj}^s \mathrm{sh}\, r_{nj}[x-(s-1)D] + Q_{nj}^s \mathrm{sh}\, r_{nj}(x-sD)\right\} \cos(a_n y)\sin(c_j z),$$

$$\psi_3^s = \sum_{n+j>0}^{\infty} \left\{W_{nj}^s \mathrm{sh}\, r_{nj}[x-(s-1)D] + Y_{nj}^s \mathrm{sh}\, r_{nj}(x-sD)\right\} \sin(a_n y)\cos(c_j z),$$

$$x > (M-1)D:$$

$$\varphi^r = Te^{ik_1[x-(M-1)D]} + \sum_{n+j>0} C_{nj} e^{-q_{nj}[x-(M-1)D]} \cos(a_n y)\cos(c_j z),$$

$$\psi_1^r = \sum_{n+j>0} D_{nj}^1 e^{-r_{nj}[x-(M-1)D]} \sin(a_n y)\sin(c_j z),$$

$$\psi_2^r = \sum_{n+j>0} D_{nj}^2 e^{-r_{nj}[x-(M-1)D]} \cos(a_n y)\sin(c_j z) \qquad (2c)$$

$$\psi_3^r = \sum_{n+j>0} D_{nj}^3 e^{-r_{nj}[x-(M-1)D]} \sin(a_n y)\cos(c_j z),$$

where

$$q_{nj} = \sqrt{a_n^2 + c_j^2 - k_1^2}, \quad r_{nj} = \sqrt{a_n^2 + c_j^2 - k_2^2}, \quad a_n = \frac{\pi n}{a}, \quad c_j = \frac{\pi j}{c}.$$

Here all capital letters are some unknown constants, k_1, k_2 are the longitudinal and the transverse wave numbers: $k_1 = \omega / c_p$ and $k_2 = \omega / c_s$, c_p and c_s are the longitudinal and the transverse wave speeds in the material $(c_p > c_s)$, R and T are the reflection and the transmission coefficients, respectively.

Let us restrict the consideration to the one-mode case: $0 < k_2 a < \pi$, $0 < k_2 c < \pi$, then $q_{nj} > 0, r_{nj} > 0$ for all $n + j = 1,2,\ldots$. For $n = j = 0, q_{00} = -ik_1$ and $r_{00} = -ik_2$, according to the radiation condition. Besides, we assume that the planes, containing arrays of cracks, are sufficiently distant from each other.

The components of the stress tensor $\sigma_{xx}, \sigma_{xy}, \sigma_{xz}$ and the displacement field u_x, u_y, u_z can be expressed in terms of the Lame wave potentials in standard form. The potentials $\psi(y,z)$ should be considered with the additional condition:

$$\frac{\partial \psi_1}{\partial x} + \frac{\partial \psi_2}{\partial y} + \frac{\partial \psi_3}{\partial z} = 0. \qquad (3)$$

Accepting the continuity of the displacement field u_x, u_y, u_z outside crack's domain, we introduce the following new unknown functions $\underline{g}^s = (g_x^s(y,z), g_y^s(y,z), g_z^s(y,z)),\ s = 1,...M$ as follows:

$$x = 0: \quad \underline{u}^l - \underline{u}^1 = \begin{cases} \underline{g}^1(y,z), & (y,z) \in \text{crack}, \\ 0, & (y,z) \notin \text{crack} \end{cases} \qquad (4a)$$

$$x = (s-1)D: \quad \underline{u}^{s-1} - \underline{u}^s = \begin{cases} \underline{g}^s(y,z), & (y,z) \in \text{crack}, \\ 0, & (y,z) \notin \text{crack} \end{cases} \qquad (4b)$$

$$x = (M-1)D: \quad \underline{u}^{M-1} - \underline{u}^r = \begin{cases} \underline{g}^M(y,z), & (y,z) \in \text{crack}, \\ 0, & (y,z) \notin \text{crack} \end{cases} \qquad (4c)$$

whose physical meaning is a relative displacement of the left and the right crack's faces along respective Cartesian direction.

Now Lame expressions for the components of the displacement field together with Eqs.(4) can be used to represent expressions for all constants appeared in potentials (1),(2) in terms of $g_x^s(y,z)$, $g_y^s(y,z)$, $g_z^s(y,z),\ s = 1,...M$. It can be proved, by analogy to the simpler 2d problem [11], that, due to the natural geometric symmetry of the problem, the relative tangential shifts between crack's faces are identically trivial:

$$g_y^s(y,z) \equiv 0, \ g_z^s(y,z) \equiv 0, \ s = 1,...M. \qquad (5)$$

Then the orthogonality of the trigonometric functions reduces Eqs. (4) to the following relations:

$$(A_{nj}+H_{nj}\mathrm{sh}(q_{nj}D))q_{nj}+(B_{nj}^{3}+Y_{nj}^{1}\mathrm{sh}(r_{nj}D))a_n-$$
$$-(B_{nj}^{2}+Q_{nj}^{1}sh(r_{nj}D))c_j=\frac{2\delta_{nj}}{ac}\int_{S_0}g_x^1(\eta,\zeta)\cos(a_n\eta)\cos(c_j\zeta)\,d\eta d\zeta, \qquad (6a)$$

$$(F_{nj}^{s-1}+H_{nj}^{s})\mathrm{sh}(q_{nj}D)q_{nj}+(W_{nj}^{s-1}+Y_{nj}^{s})\mathrm{sh}(r_{nj}D)a_n-$$
$$-(V_{nj}^{s-1}+Q_{nj}^{s})\mathrm{sh}(r_{nj}D)c_j=\frac{2\delta_{nj}}{ac}\int_{S_0}g_x^s(\eta,\zeta)\cos(a_n\eta)\cos(c_j\zeta)\,d\eta d\zeta, \qquad (6b)$$

$$(F_{nj}^{M-1}\mathrm{sh}(q_{nj}D)+C_{nj})q_{nj}+(W_{nj}^{M-1}\mathrm{sh}(r_{nj}D)-D_{nj}^{3})a_n-$$
$$-(V_{nj}^{M-1}\mathrm{sh}(r_{nj}D)-D_{nj}^{2})c_j=\frac{2\delta_{nj}}{ac}\int_{S_0}g_x^M(\eta,\zeta)\cos(a_n\eta)\cos(c_j\zeta)\,d\eta d\zeta, \qquad (6c)$$

$$-ik_1R-H_0^1k_1\sin(k_1D)=\frac{1}{4ac}\int_{S_0}g_x^1(\eta,\zeta)\,d\eta d\zeta, \qquad (7a)$$

$$-F_0^{s-1}k_1\sin(k_1D)-H_0^sk_1\sin(k_1D)=\frac{1}{4ac}\int_{S_0}g_x^s(\eta,\zeta)\,d\eta d\zeta, \qquad (7b)$$

$$ik_1e^{ik_1(M-1)D}-F_0^{M-1}k_1\sin(k_1D)-ik_1T=\frac{1}{4ac}\int_{S_0}g_x^M(\eta,\zeta)\,d\eta d\zeta, \qquad (7c)$$

where

$$\delta_{nj}=\begin{cases}1/2, & (n,j)=1,2...\\ 1/4, & (n=0,j=1,2...);\ (j=0,n=1,2...).\end{cases}$$

The boundary conditions for the stress and the displacement fields over planes $x=(s-1)D,\ s=1,...,M$ are as follows:

$$\sigma_{xx}^{l}=\sigma_{xx}^{1},\ \sigma_{xx}^{s-1}=\sigma_{xx}^{s},\ \sigma_{xx}^{M-1}=\sigma_{xx}^{r},\ (y,z)\notin\text{crack}, \qquad (8a)$$

$$\sigma_{xx}^{l}=\sigma_{xx}^{1}=0,\ \sigma_{xx}^{s-1}=\sigma_{xx}^{s}=0,\ \sigma_{xx}^{M-1}=\sigma_{xx}^{r}=0,\ (y,z)\in\text{crack}, \qquad (8b)$$

$$u_x^{M-1}=u_x^{r},\quad u_x^{s-1}=u_x^{s},\quad u_x^{M-1}=u_x^{r},\quad (y,z)\notin\text{crack}, \qquad (8c)$$

where relations (8a) mean the continuity of the stress field and relations (8c) the continuity of the displacements, all -- outside the crack. The conditions for the stress σ_{xy},σ_{xz} and the displacement fields u_y,u_z have analogous forms. Now, by substituting all the constants into the boundary conditions (8) of the stress-free faces of the cracks for $x=0,D,2D,...(M-1)D$, $(y,z)\in S_0$, with the use of the basic assumption $D/a>>1, D/c>>1$, after some routine transformations (see Appendix A in [26]) one obtains the following main system of integral equations ($(y,z)\in S_0$) :

$$\frac{1}{ac}\int_{S_0} g_x^1(\eta,\zeta)\{\frac{1}{8ik_1}-\frac{K(y-\eta,z-\zeta)}{k_2^4}\}d\eta d\zeta+\frac{e^{ik_1 D}}{8acik_1}\int_{S_0} g_x^2(\eta,\zeta)d\eta d\zeta+$$

$$+\frac{e^{ik_1 2D}}{8acik_1}\int_{S_0} g_x^3(\eta,\zeta)d\eta d\zeta+...+\frac{e^{ik_1(M-1)D}}{8acik_1}\int_{S_0} g_x^M(\eta,\zeta)d\eta d\zeta=1, \qquad (9a)$$

$$\frac{e^{ik_1 D}}{8acik_1}\int_{S_0} g_x^1(\eta,\zeta)d\eta d\zeta+\frac{1}{ac}\int_{S_0} g_x^2(\eta,\zeta)\{\frac{1}{8ik_1}-\frac{K(y-\eta,z-\zeta)}{k_2^4}\}d\eta d\zeta+$$

$$+\frac{e^{ik_1 D}}{8acik_1}\int_{S_0} g_x^3(\eta,\zeta)d\eta d\zeta+...+\frac{e^{ik_1(M-2)D}}{8acik_1}\int_{S_0} g_x^M(\eta,\zeta)d\eta d\zeta=e^{ik_1 D}, \qquad (9b)$$

$$\frac{e^{ik_1 2D}}{8acik_1}\int_{S_0} g_x^1(\eta,\zeta)d\eta d\zeta+\frac{e^{ik_1 1D}}{8acik_1}\int_{S_0} g_x^2(\eta,\zeta)d\eta d\zeta+\frac{1}{ac}\int_{S_0} g_x^3(\eta,\zeta)\{\frac{1}{8ik_1}-$$

$$-\frac{K(y-\eta,z-\zeta)}{k_2^4}\}d\eta d\zeta+...+\frac{e^{ik_1(M-3)D}}{8acik_1}\int_{S_0} g_x^M(\eta,\zeta)d\eta d\zeta=e^{ik_1 2D}, \qquad (9c)$$

...

$$\frac{e^{ik_1(M-1)D}}{8acik_1}\int_{S_0} g_x^1(\eta,\zeta)d\eta d\zeta+\frac{e^{ik_1(M-2)D}}{8acik_1}\int_{S_0} g_x^2(\eta,\zeta)d\eta d\zeta+...+$$

$$+\frac{e^{ik_1 D}}{8acik_1}\int_{S_0} g_x^{M-1}(\eta,\zeta)d\eta d\zeta+\frac{1}{ac}\int_{S_0} g_x^M(\eta,\zeta)\{\frac{1}{8ik_1}-\frac{K(y-\eta,z-\zeta)}{k_2^4}\}d\eta d\zeta=$$

$$=e^{ik_1(M-1)D}, \qquad (9d)$$

where $K(y,z)$ and R_{nj} (the Rayleigh function) are:

$$K(y,z)=\sum_{n+j>0}\delta_{nj}\frac{R_{nj}}{q_{nj}}\cos(a_n y)\cos(c_j z);R_{nj}=[2(a_n^2+c_j^2)-k_2^2]^2-4r_{nj}q_{nj}(a_n^2+c_j^2). \qquad (10)$$

Let us consider the auxiliary integral equation:

$$\frac{1}{ack_2^2}\int_{S_0} h(\eta,\zeta)K(y-\eta,z-\zeta)d\eta d\zeta=1, \qquad (y,z)\in S_0. \qquad (11)$$

It is obvious that

$$g_x^1(y,z) = \left(\frac{1}{8acik_1}J_1 + \frac{e^{ik_1 D}}{8acik_1}J_2 + \frac{e^{ik_1 2D}}{8acik_1}J_3 + \ldots + \frac{e^{ik_1 (M-1)D}}{8acik_1}J_M - \right.$$

$$\left. -1\right)k_2^2 h(y,z), \qquad (12a)$$

$$g_x^2(y,z) = \left(\frac{e^{ik_1 D}}{8acik_1}J_1 + \frac{1}{8acik_1}J_2 + \frac{e^{ik_1 D}}{8acik_1}J_3 + \ldots + \frac{e^{ik_1 (M-2)D}}{8acik_1}J_M - \right.$$

$$\left. -e^{ik_1 D}\right)k_2^2 h(y,z), \qquad (12b)$$

$$g_x^3(y,z) = \left(\frac{e^{ik_1 2D}}{8acik_1}J_1 + \frac{e^{ik_1 D}}{8acik_1}J_2 + \frac{1}{8acik_1}J_3 + \ldots + \frac{e^{ik_1 (M-3)D}}{8acik_1}J_M - \right.$$

$$\left. -e^{ik_1 2D}\right)k_2^2 h(y,z), \qquad (12c)$$

$$\ldots$$

$$g_x^M(y,z) = \left(\frac{e^{ik_1 (M-1)D}}{8acik_1}J_1 + \frac{e^{ik_1 (M-2)D}}{8acik_1}J_2 + \ldots + \frac{e^{ik_1 D}}{8acik_1}J_{M-1} + \frac{1}{8acik_1}J_M - \right.$$

$$\left. -e^{ik_1 (M-1)D}\right)k_2^2 h(y,z), \qquad J_j = \int_{S_0} g_x^j(\eta,\zeta)\,d\eta\,d\zeta, \quad j = 1,2,\ldots,M. \qquad (12d)$$

By integrating Eqs. (12) over S_0, one obtains the system of linear algebraic equations for the unknown quantities J_j. With respect to $\widetilde{J}_j = J_j/8acik_1$, $j = 1,2,\ldots,M$ it appears with the following matrix

$$\left(\begin{array}{ccccc|c} \alpha & \beta & \beta^2 & \ldots & \beta^{M-1} & 1 \\ \beta & \alpha & \beta & \ldots & \beta^{M-2} & \beta \\ \beta^2 & \beta & \alpha & \ldots & \beta^{M-3} & \beta^2 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \beta^{M-1} & \beta^{M-2} & \beta^{M-3} & \ldots & \alpha & \beta^{M-1} \end{array}\right), \qquad (13)$$

where $\alpha = 1 - 8acik_1 / k_2^2 H$, $\beta = e^{ik_1 D}$, $H = \int_{S_0} h(\eta,\zeta)\,d\eta\,d\zeta.$

Therefore, once the auxiliary equation (11) and the system with the matrix (13) are solved, all necessary characteristics of the wave field can be found. In particular, the reflection and the transmission coefficients are defined as follows (see [26]):

$$R = -\frac{J_1}{8acik_1} - \frac{J_2}{8acik_1}e^{ik_1 D} - \ldots - \frac{J_M}{8acik_1}e^{ik_1 (M-1)D}, \qquad (14a)$$

$$T = -\frac{J_1}{8acik_1} e^{ik_1(M-1)D} - \frac{J_2}{8acik_1} e^{ik_1(M-2)D} - \ldots - \frac{J_M}{8acik_1} + e^{ik_1(M-1)D} . \quad (14b)$$

It can be shown that the natural energetic condition $|R|^2 + |T|^2 = 1$ is satisfied for any real-valued quantity H.

Numerical solution of the integral equation

To be more specific, let us restrict the consideration by the case of equal periods of the grating: $a = c$. Then the basic dual integral equation (14) can be rewritten in the following dimensionless form ($c = a = 1$):

$$\frac{1}{k_2^2} \int_{S_0} h(\eta,\zeta) \left\{ \Phi_r(y-\eta, z-\zeta) + \frac{k_2^2 - k_1^2}{2\pi[(y-\eta)^2 + (z-\zeta)^2]^{3/2}} \right\} d\eta d\zeta = 1,$$

$$\Phi_r(y,z) = -2(k_2^2 - k_1^2) I_r(y,z) + K_r(y,z), \qquad (y,z) \in S_0. \qquad (15)$$

In order to provide the stability of the numerical treatment, in the performed numerical experiments there is applied a discrete quadrature formulae, for 2-D hyper-singular kernels, known as a "method of discrete vortices" [20]. It is proved in [20] that with a discretization of Eq. (11) a stable treatment of the hyper-singular kernel of the type $1/r^{3/2}$, where $r^2 = (y-\eta)^2 + (z-\zeta)^2$, can be attained if one chooses two different meshes of nodes for the «internal» variables η, ζ and the «external» variables y, z. More precisely, if one subdivides the interval of integration $(-b, b)$ to N_1 equal small sub-intervals and the interval $(-d, d)$ to N_2 equal sub-intervals, and if the «internal» nodes over each Cartesian coordinate y and z are chosen just at the ends of the respective small sub-intervals, then the «external» nodes should be chosen every time at the middle points between two neighbor «internal» nodes:

$$\eta_k = -b + k\varepsilon_1, \qquad y_l = -b + (l - 1/2)\varepsilon_1, \qquad \varepsilon_1 = 2b / N_1,$$
$$\zeta_m = -d + m\varepsilon_2, \qquad z_p = -d + (p - 0.5)\varepsilon_2, \qquad \varepsilon_2 = 2d / N_2, \qquad (16)$$
$$k = 0, \ldots, N_1, \quad l = 1, \ldots, N_1, \qquad m = 0, \ldots, N_2, \quad p = 1, \ldots, N_2.$$

With such a treatment the discretization of Eq. (15) implies:

$$\frac{1}{k_2^2} \sum_{k,m=1}^{N} h(\eta_k, \zeta_m) \{ \varepsilon_1 \varepsilon_2 \Phi_r(y_l - \eta_k, z_p - \zeta_m) +$$

$$+ \frac{k_2^2 - k_1^2}{2\pi} \int_{\eta_{k-1}}^{\eta_k} \int_{\zeta_{m-1}}^{\zeta_m} \frac{1}{[(y_l - \eta)^2 + (z_p - \zeta)^2]^{3/2}} d\eta d\zeta \} = 1. \qquad (17)$$

Further, it is proved in [20,21] that integration of the hyper-singular kernels in Eq. (17) may be performed, by using standard antiderivatives, in the same way like in the case of usual continuous functions. Thus, using tabulated integrals, see [22], as a result, Eq. (17) is reduced to the following system of linear algebraic equations in the discrete form:

$$\frac{1}{k_2^2}\sum_{k,m=1}^{N} h(\eta_k,\zeta_m)\{\varepsilon_1\varepsilon_2\Phi_r(y_l-\eta_k, z_p-\zeta_m)+$$

$$+\frac{k_2^2-k_1^2}{2\pi}\left[-\frac{\sqrt{(\eta_k-y_l)^2+(\zeta_m-z_p)^2}}{(\eta_k-y_l)(\zeta_m-z_p)}+\frac{\sqrt{(\eta_k-y_l)^2+(\zeta_{m-1}-z_p)^2}}{(\eta_k-y_l)(\zeta_{m-1}-z_p)}+\right.$$

$$\left.\left.+\frac{\sqrt{(\eta_{k-1}-y_l)^2+(\zeta_m-z_p)^2}}{(\eta_{k-1}-y_l)(\zeta_m-z_p)}-\frac{\sqrt{(\eta_{k-1}-y_l)^2+(\zeta_{m-1}-z_p)^2}}{(\eta_{k-1}-y_l)(\zeta_{m-1}-z_p)}\right]\right\}=1. \quad (18)$$

It is proved in [20] that the applied method of discrete vortices automatically provides the required condition prescribing that crack's opening should vanish when approaching its outer boundary, the perimeter of domain S_0. In the initial continuous form, see Eqs. (9) and (11), this follows from the qualitative properties of respective hyper-singular equations, see [21]. In the discrete form this is provided by the applied numerical technique, see [20].

Some examples of the calculations are presented in figures 2 - 4, all for the quadratic unit cell ($a=c$), for a particular elastic material $c_p/c_s=1.870$. All the Figures are the behaviors of the transmission coefficient versus frequency parameter. For fixed D and M the different lines are related to different crack's sizes (Fig. 2). The behavior of the coefficient with all the parameters for Fig. 2, except M (here M is twice more), is illustrated on Fig. 3. Fig. 4 demonstrates the comparison of the behavior when M takes a pair of values.

Numerical results and physical conclusions

The obtained results are analyzed on the subject of the cutoff properties of the acoustic metamaterials possessing an internal periodic geometric structure, as described in the Introduction.

In the numerical analysis of the qualitative properties of the considered geometry of the cracks in the framework of the proposed here method will do the main emphasis on the physical properties of the system as an acoustic filter. Investigate the possibility of using the considered accurate artificial grating of cracks are made in elastic material, for the organization of cutoff frequency ranges with the passage of the plane longitudinal wave. Use for this purpose the exact calculation according to the obtained formulas (14), based on the accurate numerical solution of auxiliary integral equation (15) and calculate the value H according to the formula (13). Obviously, cutoff frequency range occurs when the value of the transmission coefficient $|T(ak_2/\pi)|$ approaching zero.

First of all, note that the cutoff frequency interval in the upper part of the single-mode frequency range $0<ak_2/\pi<1$ is achieved for any geometrical and physical parameters, example of this type of dependence on frequency is shown in Fig.2. Obviously, in both cases almost complete locking of the wave channel is achieved in the frequency range $ak_2/\pi>0.85$. However, from a practical point of view, the frequency filters are more effective, when they allow to achieve the filtering of the waves not only in the upper part of one mode frequency range or for the extremely large values of frequencies in this mode.

Detailed analysis shows that almost any desired frequency interval with a locking wave channel is achievable by control of the parameters of the relative size of cracks b/a, the number of vertical arrays M and the distance between the adjacent planes with arrays D/a. It turns out that the function $|T(b/a)|$ is smooth enough and the dependence on parameters M and D/a is more complex, where the intervals of increase and decrease follow each other. In this regard, the control wave process by changing the crack length is more effective.

It can be seen From Fig.2, where $M = 5,\ b/a = 0.5$ the frequency locking interval $0.37 < ak_2 / \pi < 0.54$ and at $b/a = 0.7$ the appropriate interval shifts to the left, reaching $0.31 < ak_2 / \pi < 0.51$. With increasing of vertical arrays ($M = 10$) the behavior of function $|T(ak_2 / \pi)|$ either demonstrates the property of acoustic filters which attained by choosing of the relative size of cracks b/a (Fig.3).

A detailed study shows that with the increase in the number of vertical arrays M, while maintaining the values of all other parameters, the cutoff frequency range is virtually unchanged. The closing process becomes more pronounced in the sense that a value in this range becomes almost constant and equal to zero and the corresponding curve is almost flat. The value of the transmission coefficient is almost zero uniformly on the whole cutoff interval. This property is demonstrated in Fig.4, where locking in the frequency interval $0.44 < ak_2 / \pi < 0.64$ for the case of $M = 10$ is more pronounced than for the case $M = 5$. It should be noted that the recent publications of the authors were devoted to a two-dimensional problem for the two parallel arrays of cracks [25] and a three-dimensional problem of wave propagation through a doubly periodic array of cracks [26].

Appendix A. Efficient treatment of the kernel

Regarding the kernel $K(y,z)$ in Eq.(11), first of all, we notice that $L_{nj} = -2(k_2^2 - k_1^2)(a_n^2 + c_j^2)^{1/2},\ (n, j) \to \infty.$ Hence, the sum defining the kernel can be transformed as follows:

$$K(y,z) = -2(k_2^2 - k_1^2) \sum_{n+j>0} \delta_{nj} (a_n^2 + c_j^2)^{1/2} \cos(a_n y)\cos(c_j z) +$$

$$+ \sum_{n+j>0} \delta_{nj} [L_{nj} + 2(k_2^2 - k_1^2)(a_n^2 + c_j^2)^{1/2}] \cos(a_n y)\cos(c_j z),$$

$$K(y,z) = -2(k_2^2 - k_1^2) I(y,z) + K_r(y,z), \tag{A1}$$

where the second term in the kernel K_r is a certain regular function. The first one itself consists of a regular and a singular part: $I(y,z) = I_r(y,z) + I_s(y,z)$. To be more specific, we further demonstrate the mathematical transformations in the particular case: $a = c$. Let us introduce the dimensionless variables $\tilde{y} = y/a, \tilde{z} = z/c$ and then, omitting tildes, one rewrites:

$$\frac{a}{\pi}I(y,z)=\sum_{n+j>0}\delta_{nj}(n^2+j^2)^{1/2}\cos(\pi ny)\cos(\pi jz)=\sum_{n=1}^{\infty}\delta_{n0}n\cos(\pi ny)+$$

$$+\sum_{j=1}^{\infty}\delta_{0j}\,j\cos(\pi jz)+\sum_{n=1}^{\infty}\sum_{j=1}^{\infty}\delta_{nj}(n^2+j^2)^{1/2}\cos(\pi ny)\cos(\pi jz)= \tag{A2}$$

$$=\frac{1}{4}\sum_{n=1}^{\infty}n\cos(\pi ny)-\frac{1}{4}\sum_{j=1}^{\infty}j\cos(\pi jz)+\frac{1}{2}\sum_{n=0}^{\infty}\sum_{j=1}^{\infty}(n^2+j^2)^{1/2}\cos(\pi ny)\cos(\pi jz)\,.$$

The last double sum can be evaluated by using the Poisson formula:

$$\sum_{n=0}^{\infty}p(n)=\frac{p(0)}{2}+P(0)+2\sum_{n=1}^{\infty}P(2\pi n), \tag{A3}$$

where in the problem at hand:

$$p(t)=(t^2+j^2)^{1/2}\cos(\pi ty),\qquad P(u)=\int_0^{\infty}p(t)\cos(ut)dt. \tag{A4}$$

By taking the generalized value of the divergent integral in $P(u)$, see for example [21], one obtains for $j\geq 1$:

$$P(u)=\lim_{\varepsilon\to+0}\int_0^{\infty}e^{-\varepsilon t}p(t)\cos(ut)dt=-\frac{j}{2}\left[\frac{K_1(j\,|\,\pi y+u\,|)}{|\,\pi y+u\,|}+\frac{K_1(j\,|\,\pi y-u\,|)}{|\,\pi y-u\,|}\right], \tag{A5}$$

where some tabulated integrals have been used [22]. Here in Eq. (A5) $K_1(\xi)$ is Macdonald's function [23]. Thus, the sum over n in Eq. (A2) takes the form ($j\geq 1$):

$$\sum_{n=0}^{\infty}(n^2+j^2)^{1/2}\cos(\pi ny)=\frac{j}{2}-\frac{jK_1(j\pi\,|\,y\,|)}{\pi\,|\,y\,|}-j\sum_{n=1}^{\infty}\left[\frac{K_1(j\,|\,2\pi n+\pi y\,|)}{|\,2\pi n+\pi y\,|}+\frac{K_1(j\,|\,2\pi n-\pi y\,|)}{|\,2\pi n-\pi y\,|}\right]. \tag{A6}$$

Finally, $I(y,z)$ in Eq. (A2) can be rewritten as follows:

$$\frac{a}{\pi}I(y,z)=\frac{1}{4}\sum_{n=1}^{\infty}n\cos(\pi ny)-\frac{1}{2\pi\,|\,y\,|}\sum_{j=1}^{\infty}jK_1(j\pi\,|\,y\,|)\cos(\pi jz)-$$

$$-\frac{1}{2}\sum_{n,j=1}^{\infty}\left[\frac{K_1(j\,|\,2\pi n+\pi y\,|)}{|\,2\pi n+\pi y\,|}+\frac{K_1(j\,|\,2\pi n-\pi y\,|)}{|\,2\pi n-\pi y\,|}\right]j\cos(\pi jz). \tag{A7}$$

The first series in the first line in Eq. (A7) can be calculated by using the generalized value of the following tabulated series, see [22]:

$$\sum_{n=1}^{\infty}n\cos(\pi ny)=\lim_{\varepsilon\to+0}\sum_{n=1}^{\infty}e^{-\varepsilon n}n\cos(\pi ny)=-\frac{1}{4\sin^2(\pi y/2)}. \tag{A8}$$

The second series in the first line in Eq. (A7) can also be calculated explicitly, by its transformation to a tabulated series, see [22]:

$$\sum_{j=1}^{\infty}jK_1(j\pi\,|\,y\,|)\cos(\pi jz)=-\frac{1}{\pi}\frac{\partial}{\partial y}\sum_{j=1}^{\infty}K_0(j\pi\,|\,y\,|)\cos(\pi jz)=$$

$$= -\frac{1}{2\pi}\frac{\partial}{\partial y}\left\{\frac{1}{(y^2+z^2)^{1/2}} + \mathbf{C} + \ln\frac{|y|}{4} + \sum_{j=1}^{\infty}\left[\frac{1}{[y^2+(2j+z)^2]^{1/2}} - \frac{1}{2j}\right] + \right.$$

$$\left. + \sum_{j=1}^{\infty}\left[\frac{1}{[y^2+(2j-z)^2]^{1/2}} - \frac{1}{2j}\right]\right\} =$$

$$= \frac{1}{2\pi}\left\{\frac{|y|}{(y^2+z^2)^{3/2}} - \frac{1}{|y|} + \sum_{j=1}^{\infty}\left[\frac{|y|}{[y^2+(2j+z)^2]^{3/2}} + \frac{|y|}{[y^2+(2j-z)^2]^{3/2}}\right]\right\}, \qquad \text{(A9)}$$

where $\mathbf{C} = 0.5772157$ is Euler's constant [23].

Therefore, one obtains for the kernel of the basic dual integral equation (11)

$$K(y,z) = K_r(y,z) - 2(k_2^2 - k_1^2)[I_r(y,z) + I_s(y,z)], \qquad \text{(A10)}$$

where the regular and the singular parts are, respectively:

$$\frac{a}{\pi}I_r = -\frac{1}{4\pi^2}\sum_{j=1}^{\infty}\left[\frac{1}{[y^2+(2j+z)^2]^{3/2}} + + \frac{1}{[y^2+(2j-z)^2]^{3/2}}\right] -$$

$$-\frac{1}{2}\sum_{n,j=1}^{\infty} j\cos(\pi jz)\left[\frac{K_1(j\,|2\pi n+\pi y|)}{|2\pi n+\pi y|} + \frac{K_1(|\,j(2\pi n-\pi y|)}{|2\pi n-\pi y|}\right] +$$

$$+\frac{1}{4\pi^2 y^2} - \frac{1}{16\sin^2(\pi y/2)}, \qquad \frac{a}{\pi}I_s = -\frac{1}{4\pi^2(y^2+z^2)^{3/2}}. \qquad \text{(A11)}$$

One can see that the obtained singular behavior of the kernel for small arguments contains a 2D hyper-singular term $1/(y^2+z^2)^{3/2}$, well known in the linear elasticity theory for cracks in unbounded media [24].

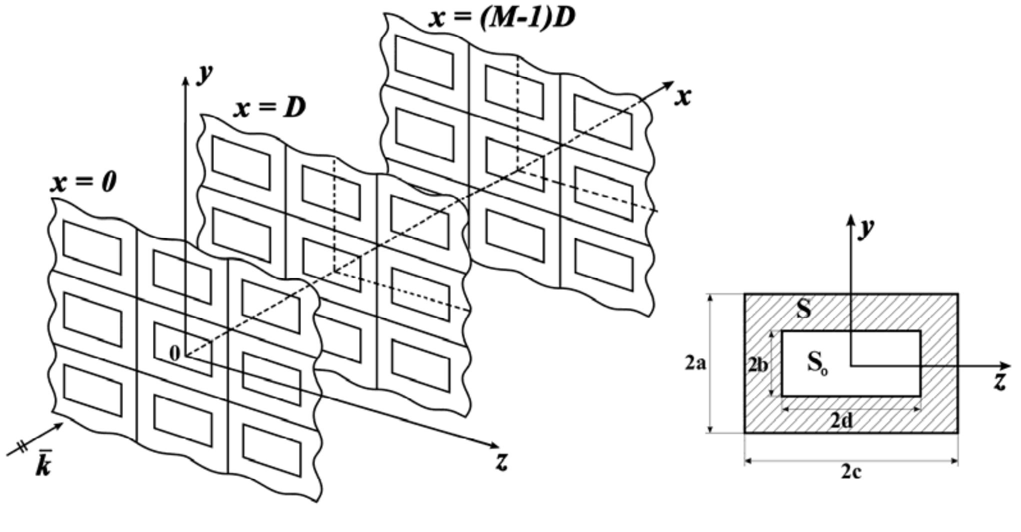

Figure 1: Propagation of the incident wave through a triple periodic array of cracks.

A unit cell.

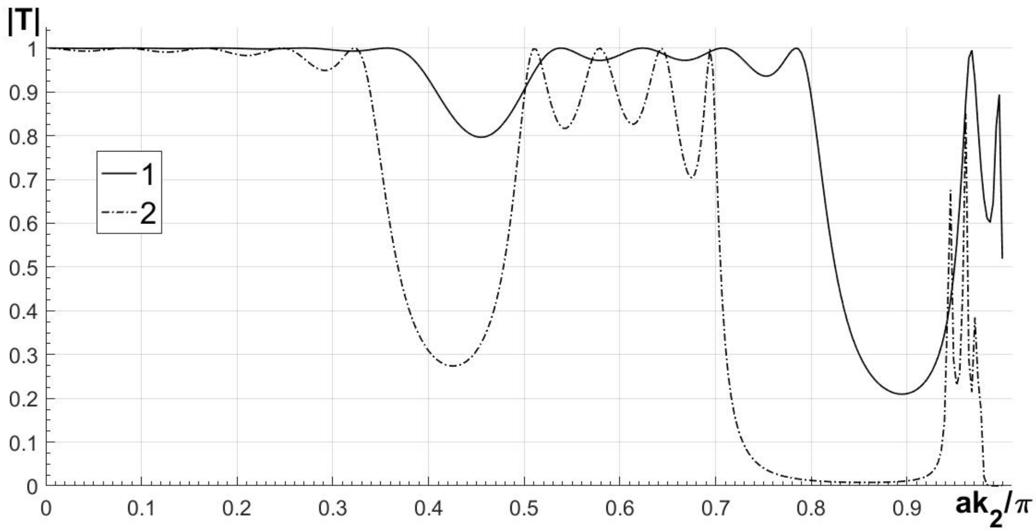

Figure 2: Transmission coefficient versus frequency parameter: b = d, a = c = 1, D/a=4, M = 5, line 1 – b/a = 0.5, line 2 – b/a = 0.7

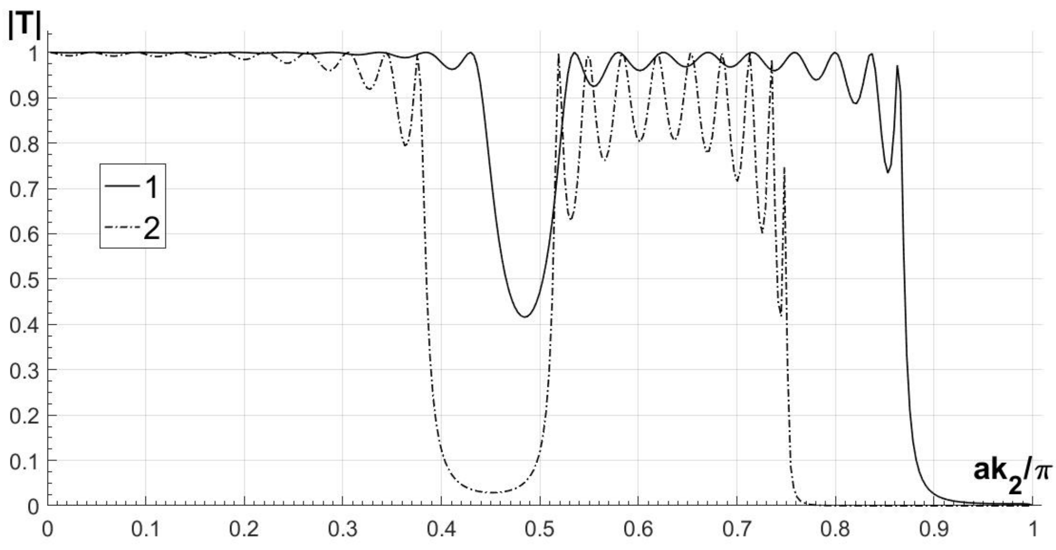

Figure 3: Transmission coefficient versus frequency parameter: b = d, a = c = 1, D/a = 4, M = 10, line 1 – b/a = 0.5, line 2 – b/a = 0.7

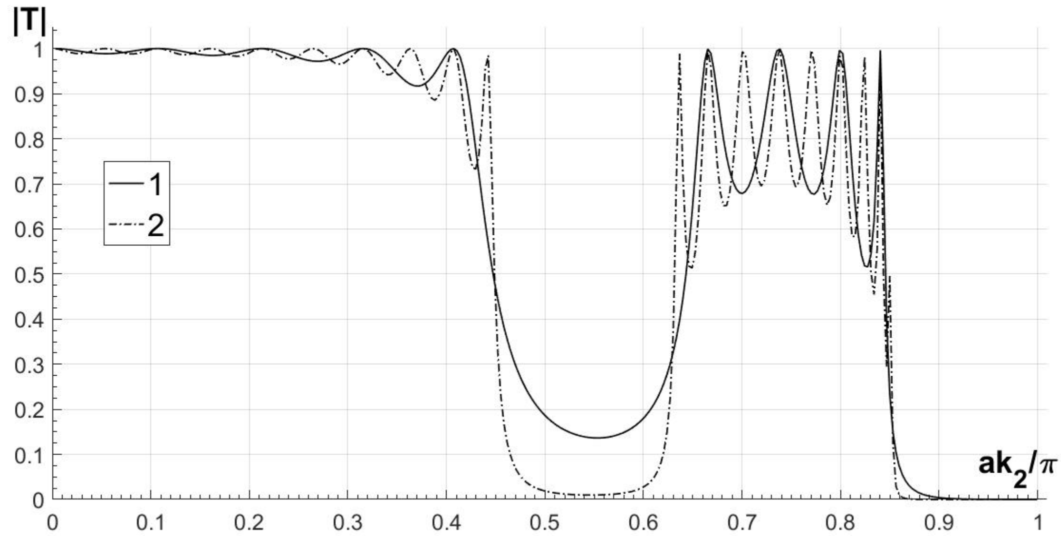

Figure 4: Transmission coefficient versus frequency parameter: b = d = 0.7, a = c = 1, D/a = 3, line 1 – M = 5, line 2 – M = 10

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Information about the authors:

Remizov M.Yu., Ph.D., Associate Professor
Affiliation: Institute of Mathematics, Mechanics and Computer Science,
Southern Federal University, **Address:** Milchakova Street 8a, 344090 Rostov-on-Don, Russia. **Tell:** +7 9034747683. **E-mail:** remizov72@mail.ru

Sumbatyan M.A., Dr. of Sciences, Professor
Affiliation: Institute of Mathematics, Mechanics and Computer Science,Southern Federal University, **Address:** Milchakova Street 8a, 344090 Rostov-on-Don, Russia
Tell: +7 9281397067. **E-mail:** sumbat@math.sfedu.ru

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