ՀԱՅԱՍՏԱՆԻ ԳԻՏՈՒԹՅՈՒՆՆԵՐԻ ԱԶԳԱՅԻՆ ԱԿԱԴԵՄԻԱՅԻ ՏԵՂԵԿԱԳԻՐ
ИЗВЕСТИЯ НАЦИОНАЛЬНОЙ АКАДЕМИИ НАУК АРМЕНИИ

Մեխանիկա 70, №2, 2017 Механика

UDC 539.3

THREE-DIMENSIONAL PROBLEM OF WAVES PROPAGATION IN HALF-SPACE WITH AN ELASTICALLY RESTRAINED BOUNDARY

Sarkisyan S.V.

Key words: elastic half-space, surface wave, elastically restrained boundary

Ключевые слова: упругое полупространство, поверхностная волна, упруго-стеснённая граница

Բանալի բառեր. առաձգական կիսատարածություն, մակերևութային ալիքներ, առաձգականորեն կաշկանդված եզր

Саркисян С.В.

Трёхмерная задача о распространении волн в полупространстве с упруго-стеснённой границей

В работе получены дисперсионные уравнения трёхмерной задачи о распространении волн в полупространстве с упруго-стеснённой границей. Исследование задачи упрощается введением потенциальных функций. Показано, что при плоской деформации упругое стеснение границы полупространства приводит к уменьшению степени локализации поверхностной волны. Трёхмерная поверхностная волна существует лишь для двух видов граничных условий, когда поверхность полупространства свободна от напряжений и стеснённая свободная поверхность. В случае стеснённой свободной поверхности трёхмерная поверхностная волна обладает свойством дисперсии.

Սարգսյան Ս.Վ.

Եռաչափ դրվածքով առաձգականորեն կաշկանդված եզրով կիսատարածությունում ալիքների տարածման խնդիրը

Եռաչափ դրվածքով պոտենցիալ ֆունկցիաների ներմուծմամբ դիտարկված է առաձգականորեն կաշկանդված եզրով կիսատարածությունում մակերևութային ալիքների գոյության հարցը: Ստացված են մակերևութային ալիքների փուլային արագության նկատմամբ բնութագրիչ հավասարումներ: Ցույց է տրված, որ հարթ դեֆորմացիայի դեպքում եզրային պայմանը բերում է մակերևութային ալիքի տեղայնացման աստիճանի նվազմանը: Եռաչափ մակերևութային ալիքները գոյություն ունեն միայն երկու դեպքում՝ կիսատարածության եզրը ազատ է լարումներից կամ կաշկանդված ազատ եզրով կիսատարածություն: Կաշկանդված եզրով կիսատարածությունում եռաչափ մակերևութային ալիքները օժտված են դիսպերսիայով:

In this paper we obtain the dispersion equation of three-dimensional wave propagation problem in half-space with an elastic- restrained border. Research of the problem is simplified by the introduction of potential functions. It is shown that by plane strain the elastic half-space constraint of boundary leads to decreasing of the surface wave localization degree. Three-dimensional surface wave exists only for two kinds of boundary conditions, when the surface of half-space is free from the stresses and the free surface is restrained. In the case of constrained free surface the three-dimensional surface wave has a dispersion property.

Introduction. Surface waves propagation study represents a separate research in science. In the study of surface waves the plane and antiplane deformation was generally considered. For the first time the existence of surface waves was indicated by Rayleigh [1], where he examined the plane problem for half-space with stress free from the boundary. Solution of the three-dimensional problem was obtained by Knowles [2], who generalized the Rayleigh problem. These results are mentioned in monograph [3]. Another option of space problem was investigated in [4]. In work [5] the three-dimensional problems for elastic space waves propagation in isotropic half-space with two options of half-space boundary conditions was researched: free boundary and when we have one shear displacement at the border of half-space, one of the tangential stress and normal stress is equal to zero. In monograph [6] the summary of elastic waves propagation space problems is given. Study of three-dimensional surface waves for various types of mixed boundary conditions on the surface of the half-space is given in work [7]. It is shown that dispersion equation has a root for two types of boundary conditions: free surface and the surface, where displacements in one tangential direction are forbidden.

Unlike the classical Rayleigh' problem, M.V. Belubekyan [8] considers two types of complex boundary conditions instead of free surface boundary conditions for an isotropic elastic half-space. It is assumed that either normal stress is constricted in the perpendicular direction to the surface normal and shear is equal to zero, or the normal stress is equal to zero and tangent is restrainedly.The conditions are set, at which the surface wave cannot exists. The problem of periodic waves propagation in an elastic layer when in the layer boundaries the normal and shear stresses restrained were investigated in works [9,10]. Here the influence of restraint factor to the phase velocity of the symmetric and asymmetric vibration layer is shown.

As a result of the integral Radon' transformation [11] the space problems of the dynamic theory of elasticity are reduced to the plane problem regarding the Radon' transformation images. In the work [12] the introduction problem of dynamic potential for solving three-dimensional problems of the dynamic theory of elasticity is investigated, in which the antiplane displacementis not used (for example, in the problem of the dynamics of the surface of an elastic half-space, where the contribution of the surface wave is dominant). Applying the Radon' transformation, the solution of three-dimensional elasticity problem is comes to solving the corresponding plane problem. Development of asymptotic models of Rayleigh' surface waves, Stoneley' and Scholte-Gogoladze' interface waves were studied in work [13]. In work [14] the wave propagation problem in an elastic half-space is studied, when the half-restrained free edge conditions on the half-space boundary are given. Using the Radon' integral transformation, a dispersion equation for determining the velocity of surface wave propagation is obtained and the numerical experiment for the different physical and mechanical parameters characterizing the media is made.

In this paper we obtain the dispersion equation of three-dimensional wave propagation problem in half-space with an elastic- restrained border. Research of the problem is simplified by the introduction of potential functions like plane strain problems [3,5]. It is shown that by plane strain the elastic half-space constraint of boundary leads to decreasing of the surface wave localization degree. Three-dimensional surface wave exists only for two kinds of boundary conditions, when the surface of half-space is free from the stresses and the free surface is restrained. In the case of constrained free surface the three-dimensional surface wave has a dispersion property. By mixed boundary conditions at the surface, the propagation angle affects the phase velocity to the three-dimensional surface wave.

Statement of the Problem. Consider the harmonic vibrations of an isotropic elastic half-space $-\infty < x < \infty$, $-\infty < z < \infty$, $0 \le y < \infty$. Vibrations described by three-dimensional motion equations [3]:

$$(\lambda + \mu)\,\mathrm{grad}\,\mathrm{div}\,\vec{u} + \mu\Delta\vec{u} = \rho\ddot{\vec{u}} \tag{1}$$

where $\vec{u}$ – displacement vector, λ, μ – Lame' parameter, ρ – density.

Suppose that the following boundary conditions are given [8] on the boundary of the half-space $y = 0$:

$$\sigma_{yx} = \alpha_* u, \;\; \sigma_{yy} = \beta_* \mathrm{v}, \;\; \sigma_{yz} = \gamma_* w\left(\alpha_*, \beta_*, \gamma_* > 0\right) \tag{2}$$

These conditions were proposed by Mindlin [16] for study the elastic wave reflections problem from the boundary of the half-space. In work [8] the conditions for the existence of Rayleigh waves in the case of elastic- restrained boundary (plane strain) were researched by M. Belubekyan. Periodic waves propagation in elastic layer is studied in works [9,10].

In particular case by $\alpha_* = \beta_* = \gamma_* = 0$ we get conditions of free boundary.

To solve the problem of surface waves propagation, the potential functions $\varphi(x, y, z, t)$

and $\psi(x, y, z, t)$ [5] are introduced like in problems of plane strain:

$$u = \frac{\partial\varphi}{\partial x} - \frac{\partial\psi}{\partial z}, \qquad w = \frac{\partial\varphi}{\partial z} + \frac{\partial\psi}{\partial x}. \tag{3}$$

By means of (1) and (3) with taking into accounts damping conditions

$$\lim_{y\to\infty} \vec{u} = 0, \lim_{y\to\infty} \varphi = 0, \lim_{y\to\infty} \psi = 0$$

the displacements u, v, w are determined in the form [5]:

$$u(x, y, z, t) = -i\begin{bmatrix} Ak\cos\gamma e^{-\upsilon_1 ky} + \\ +(Bk\cos\gamma + Ck\sin\gamma)e^{-\upsilon_2 ky} \end{bmatrix} \exp i(\omega t - xk\cos\gamma - zk\sin\gamma),$$

$$\mathrm{v}(x, y, z, t) = -k\left[A\upsilon_1 e^{-\upsilon_1 ky} + B\upsilon^{-1}{}_2 e^{-\upsilon_2 ky}\right]\exp i(\omega t - xk\cos\gamma - zk\sin\gamma), \tag{4}$$

$$w(x, y, z, t) = -i\begin{bmatrix} Ak\sin\gamma e^{-\upsilon_1 ky} + \\ +(Bk\sin\gamma - Ck\cos\gamma)e^{-\upsilon_2 ky} \end{bmatrix} \exp i(\omega t - xk\cos\gamma - zk\sin\gamma),$$

where k – wave number, $\nu_1^2 = 1 - \theta\eta$, $\nu_2^2 = 1 - \eta$, $\theta = \frac{c_t^2}{c_l^2} < 1$, $\eta = \frac{\omega^2}{k^2 c_t^2} < 1$ – dimensionless phase velocity of the three-dimensional surface wave, γ – sharp angle of wave propagation in plane $0xz$, A, B and C – arbitrary constants.

Applying Hooke's law the boundary conditions (2) by y=0 comes to the form:

$$\mu\left(\frac{\partial u}{\partial y} + \frac{\partial \mathrm{v}}{\partial x}\right) - \alpha_* u = 0, \mu\left(\frac{\partial \mathrm{v}}{\partial z} + \frac{\partial w}{\partial y}\right) - \gamma_* w = 0,$$

$$\lambda\left(\frac{\partial u}{\partial x} + \frac{\partial \mathrm{v}}{\partial y} + \frac{\partial w}{\partial z}\right) + 2\mu\frac{\partial \mathrm{v}}{\partial y} - \beta_* \mathrm{v} = 0 \tag{5}$$

Dispersion equations and numerical results. Satisfying solution (4) to the boundary conditions (5) we get the dispersion equation:

$$
\begin{aligned}
&4\nu_1\nu_2-(2-\eta)^2+\nu_2\left[\eta\left(\alpha_0+\beta_0\nu_1\nu_2^{-1}\right)+\alpha_0\beta_0\left(\nu_2^{-1}-\nu_1\right)\right]+\\
&+\gamma_0\left[4\nu_1+\eta\left(\alpha_0+\beta_0\nu_1\nu_2^{-1}\right)+\alpha_0\beta_0\left(\nu_2^{-1}-\nu_1\right)-(2-\eta)^2\nu_2^{-1}\right]+\\
&+\mathrm{tg}^2\gamma\Big[4\nu_1\nu_2-(2-\eta)^2+\nu_2\eta\left(\gamma_0+\beta_0\nu_1\nu_2^{-1}\right)+\nu_2\gamma_0\beta_0\left(\nu_2^{-1}-\nu_1\right)+\\
&+\alpha_0\left(4\nu_1-(2-\eta)^2\nu_2^{-1}+\eta\left(\gamma_0+\beta_0\nu_1\nu_2^{-1}\right)+\gamma_0\beta_0\left(\nu_2^{-1}-\nu_1\right)\right)\Big]=0,
\end{aligned}
\tag{6}
$$

where $\alpha_0=\dfrac{\alpha_*}{\mu k},\ \beta_0=\dfrac{\beta_*}{\mu k},\ \gamma_0=\dfrac{\gamma_*}{\mu k}.$

From equation (6) is follows that three-dimensional surface wave possesses dispersion feature. For given equation let's consider the following particular cases.

- Dispersion equation (6) in case of plane strain comes to the following $(\gamma=\gamma_0=0)$

$$
\begin{aligned}
&(2-\eta)^2-4\sqrt{(1-\eta)(1-\eta\theta)}-\alpha_0\eta\sqrt{1-\eta}-\beta_0\eta\sqrt{1-\eta\theta}-\\
&-\alpha_0\beta_0\left(1-\sqrt{(1-\eta)(1-\eta\theta)}\right)=0.
\end{aligned}
\tag{7}
$$

Equation (7) by $\alpha_0=\beta_0=0$ is coincides with Rayleigh' classical equation. Compared with the Rayleigh's equation, the equation (7) is dispersion, since solution depends on α_0,β_0. Dispersion equation (7) by $\alpha_0=0$ either $\beta_0=0$ has been received in work [8], where the conditions of existence of surface waves were set, depending on the coefficient characterizing the elastic restraint and the wave length.

Equation (7) has a root $\eta=0$, to which the trivial solution is corresponds. Following to work [15], eliminating root $\eta=0$, the equation (7) comes to the following:

$$
\begin{aligned}
&D(\eta)\equiv\eta-\frac{(1-\theta)\sqrt{1-\eta}}{\sqrt{1-\eta}+\sqrt{1-\theta\eta}}\left(4-\alpha_0\beta_0\right)-\\
&-\alpha_0\sqrt{1-\eta}-\beta_0\sqrt{1-\theta\eta}-\alpha_0\beta_0=0.
\end{aligned}
\tag{8}
$$

Function $D(\eta)$ takes the following values:

$$D(0) = -0.5(1-\theta)(4-\alpha_0\beta_0) - \alpha_0 - \beta_0 - \alpha_0\beta_0,$$
$$D(1) = 1 - \beta_0\sqrt{1-\theta} - \alpha_0\beta_0.$$

Equation (8) will have solution in the interval $\eta \in (0,1)$, if $D(0) < 0, \quad D(1) > 0$ and this solution will be unique if $\dfrac{dD}{d\eta} > 0.$ Choosing values α_0 and β_0, which satisfy to the given conditions, we can get the values for surface wave phase velocity depending on the degree of restraint surface of the half-space.

In table 1 the numerical results are given, which calculated by equation (8) for η parameter, characterizing the square of phase velocity of space wave depending on parameters, characterizing degree of restraint surface of the half-space by $\theta = 0.33$. The table shows that the restraint boundaries either at the direction of the normal or tangential direction leads to an increase in the dimensionless phase velocity of the surface wave. By the oppression of the border at the same time in both directions of the surface wave phase velocity at first increases, reaching a maximum value, then decreases. Thus, the boundary elastic half-space constraint reduces the degree of surface wave of localization (slow decay of the amplitude).

- Dispersion equation (6) for three-dimensional problem in case of free boundary $(\alpha_0 = \beta_0 = \gamma_0 = 0)$ comes to the following:

$$\left(1+\mathrm{tg}^2\gamma\right)\left((2-\eta)^2 - 4\sqrt{(1-\eta)(1-\eta\theta)}\right) = 0 \qquad (9)$$

By different mixed boundary conditions [7] we get corresponding equations from dispersion equation (6):

a) displacement is prohibited in tangent direction

$$\sigma_{yy} = 0, \sigma_{yz} = 0, u = 0$$

$$(2-\eta)^2 - 4\sqrt{(1-\eta)(1-\eta\theta)} - \eta(1-\eta)\,\mathrm{ctg}^2\gamma = 0 \qquad (10)$$

b) displacement is prohibited in one of tangent directions

$$\sigma_{yy} = 0,\ \sigma_{yx} = 0,\ w = 0,\ (2-\eta)^2 - 4\sqrt{(1-\eta)(1-\eta\theta)} - \eta(1-\eta)\,\mathrm{tg}^2\gamma = 0 \qquad (11)$$

Table 1

α_0	β_0	η
0	0	0.8464
0	0.2	0.9005
0	0.4	0.9405
0	0.6	0.9686
0	0.8	0.9867
0	1	0.9966
0.2	0	0.8712
0.4	0	0.8903
0.6	0	0.9052
0.8	0	0.9172
1	0	0.9269
0.2	0.2	0.9045
0.4	0.4	0.9238
0.6	0.6	0.9216
0.8	0.8	0.9001
1	1	0.8540

c) displacement is prohibited in both tangent directions:

$$\sigma_{yy} = 0, \;\; u = 0, \;\; w = 0, \; \eta\left(1 + \mathrm{t}\, g^2\gamma\right) = 0$$

d) displacement is prohibited in normal direction:

$$\mathrm{v} = 0, \;\; \sigma_{yx} = 0, \;\; \sigma_{yz} = 0, \; \eta\sqrt{1 - \theta\eta}\left(1 + \mathrm{tg}^2\gamma\right) = 0$$

e) displacement is prohibited in one of tangent directions and in normal direction

$$\mathrm{v}=0,\ w=0,\ \sigma_{yx}=0,\ \eta\sqrt{1-\theta\eta}+\mathrm{tg}^2\gamma\sqrt{1-\eta}\left(1-\sqrt{(1-\eta)(1-\theta\eta)}\right)=0$$

f) displacement is prohibited in one of tangent directions and in normal direction

$$\mathrm{v}=0,\ u=0,\ \sigma_{yz}=0,\ \eta\sqrt{1-\theta\eta}+\mathrm{ctg}^2\gamma\sqrt{1-\eta}\left(1-\sqrt{(1-\eta)(1-\theta\eta)}\right)=0$$

g) displacement is prohibited in all directions

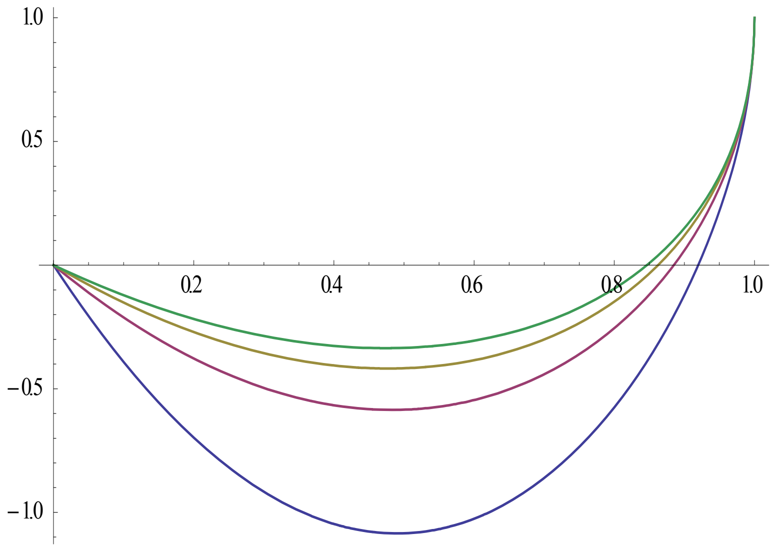
$$u=0,\ \mathrm{v}=0,\ w=0,\ \left(1+\mathrm{tg}^2\gamma\right)\left(1-\sqrt{(1-\eta)(1-\theta\eta)}\right)=0$$

Studies have shown that three-dimensional surface wave exist only for two kinds of boundary conditions. In the case where the half-surface is free from stresses, the known Rayleigh equation is obtained (9). If we have restrained free surface (displacement if forbidden in tangential direction) dispersion equations are comes to the equation (10) or (11). In case of free surface constrained these equations have a single root $\eta<1$ and three-dimensional surface wave possesses the feature of dispersion. The figure shows dependence of the phase velocity of the three-dimensional surface wave from the propagation angle. In table 2 the values of η parameter are shown, which characterized the square of phase velocity of three-dimensional surface wave depends on propagation angle by θ =0.33.

Table 2 and graph show that by mixed boundary conditions on the surface, and propagation angle affects to the phase velocity of the three-dimensional surface wave. By the displacement prohibition in one of tangential direction, with an increase of the angle value, the three-dimensional surface wave phase velocity decreases (increases). By $\gamma=\frac{\pi}{2}(0)$ the value of the phase velocity of the three-dimensional surface wave is exactly coincides to the value of the phase velocity of the Rayleigh surface waves.

Table 2

γ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
η(10)	-	0.9198	0.8850	0.8624	0.8464
(11)	0.8464	0.8624	0.8850	0.9198	-



Conclusion. Dispersion equations for the space problem of wave propagation in half-space with an elastic-restrained border are obtained. The elastic half-space restraint of boundary in plane strain leads to decreasing of localization degree of the surface wave (to the slow decay of the amplitude). Three-dimensional surface wave exists only for two kinds of boundary conditions - when the surface of half-space is free from the stresses and in case of cramped free surface (displacement is prohibitedin one of tangential direction). In case of cramped free surface three-dimensional surface wave possesses the feature of dispersion. By mixed boundary conditions on the surface, the angle of propagation has the influence to the phase velocity of three-dimensional surface wave.Increasing this angle the values of three-dimensional surface wave phase velocity decreases (increases), tends to the value of the phase velocity of the Rayleigh surface waves.

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About author:

Sarkisyan Samvel Vladimir – Doctor of Physical and Mathematical Sciences, Professor, Head of Mechanics Department, Faculty of Mathematics and Mechanics, Yerevan State University
Tel.: (+37455) 73 13 13; **E-mail:**vas@ysu.am

Received 09.12.2016