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ON A CLASS OF CONTACT PROBLEMS OF ELASTICITY THEORY, SOLVABLE BY THE INTEGRAL EQUATIONS METHOD

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Keywords: contact problems, stresses, elastic layer, half-space, wedge, integral equations.
Ключевые слова: контактные задачи, напряжения, упругий слой, полупространство, клин, интегральные уравнения.
Բանալի բառեր՝ կոնտակտային խնդիրներ, լարումներ, շերտ, կիսատարածություն, սեպ, ինտեգրալ հավասարում

Mkhitaryan S.M., Melik-Adamyan P.E.
On a Class of Contact Problems of Elasticity Theory, Solvable by the Integral Equations Method

Cosely connected with classical contact problems a certain class of contact problems of mathematical elasticity theory solvable by the method of integral equations is considered. An elastic layer, wedge and half-space under the anti-plane deformation are taken as bases.

Մխիթարյան Ս.Մ., Մելիք-Ադամյան Պ.Է.
Ինտեգրալ հավասարումների մեթոդով լուծվող առաձգականությա ն տեսության կոնտակտային խնդիրների մի դասի մասին

Դիտարկված է դասական կոնտակտային խնդիրների հետ սերտորեն կապված և ինտեգրալ հավասարումների մեթոդով լուծվող կոնտակտային խնդիրների մի դաս: Որպես հիմքեր վերցված են առաձգական շերտը, սեպը և կիսատարածությունը հակահարթ դեֆորմացիայի պայմաններում:

Мхитарян С.М., Мелик-Адамян П.Э.
Об одном классе контактных задач теории упругости, решаемых методом интегральных уравнений

Рассмотрен класс контактных задач математической теории упругости, решаемых методом интегральных уравнений, тесно связанный с классическими контактными задачами. В качестве оснований выбраны упругий слой, клин и полупространство при антиплоской деформации.

Introduction. Classical and non-classical contact and mixed boundary value problems (b.v.p.) cover a wide area of the mechanics of a deformable solid. The literature devoted to development of effective mathematical methods for their investigations and to their applications in engineering is extensive. On this subject we refer to [1]-[8] and references therein.

To provide the contact durability and rigidity of various machinery and engineering constructions it is essential the action of admissible and theoretically acceptable contact stresses or displacements in the contact zone. In the contact of two deformable solids fastened together it can be attained as by an appropriate choice of their geometrical and physical parameters, and of the outer load as well.

In the present paper a certain class of contact problems is discussed in the simplest case of anti-plane deformation. When the contact zone is under the force of pre-assigned regime of displacements, contact interactions between an elastic prismatic bar with a rectangular cross-section and an elastic layer, or a half-space, or an elastic wedge-like solid are considered. With the help of preliminary solutions of auxiliary problems, the setup problems are reduced to the Fredholm integral equations (i.e.) of the first kind with the symmetric kernels. The solutions of these i.e. are built by both the Krein method [9], [10] and the method of integral spectral relationships, established in [11]. Some special cases are presented.

1. The setting of contact problems and derivation of the main equations.

Consider two elastic solids B_1 and B_2 with given modules of elasticity and Poisson ratios $E_j,\ \nu_j\ \ (j=1,2)$ respectively, rigidly fastened by a surface S. Generally B_2 is massive deformable solid – the base of different geometrical forms, while B_1 is thin-walled element such as stringers, beams, plates and shells, which are convenient means for transmitting loading to the bases, that usually occur in the engineering practice.

Let the composite solid be a subject to outer loads (Fig.1).

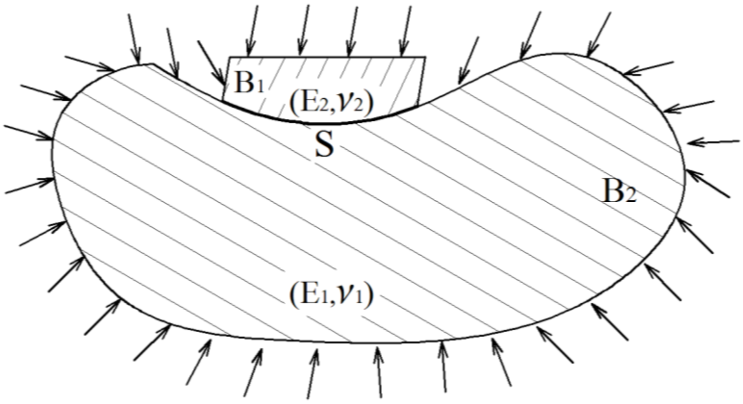

Fig.1

The main investigations in the theory of contact problems are reduced to the study of the stress-strain state of such a composite solid, especially to determining the distribution of contact stresses on S and their kinematic parameters (coming together, relative turns), characterizing the rigidity of a contact.

These problems admit the following settings.

Determine contact stresses on a contact surface S and kinematic parameters under a given surface load.

Find the outer load distribution on a surface of solid B_1 to get the pre-assigned contact stresses, admissible in theory.

Find the outer load distribution on a surface of solid B_1 to get the pre-assigned regime of displacements on the contact surface S.

The present paper concerns to the last topic. The anti-plane deformation of elastic solids in contact is assumed, and the following patterns (models) with corresponding problems are discussed.

I. In the rectangular coordinate system $Oxyz$ consider an elastic layer $\Omega = \{-\infty < x, z < \infty;\ -H \le y \le 0\}$ of the height H and shear module G (Fig.2).

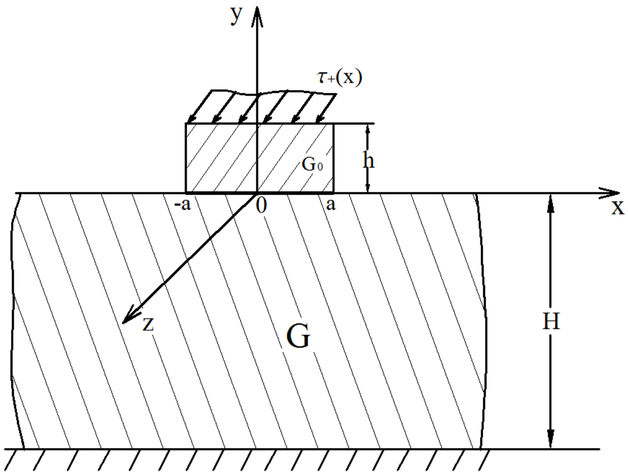

Fig.2

Let it be rigidly clamped by its lower side $y = -H$, on its boundary plane $y = 0$, be rigidly fastened by the strip $\omega = \{-a \le x \le a;\ y = 0;\ -\infty < z < \infty\}$ to the infinite prismatic bar $\Omega_0 = \{-a \le x \le a;\ 0 \le y \le h;\ -\infty < z < \infty\}$ with a rectangular cross-section $D_0 = \{-a \le x \le a;\ 0 \le y \le h\}$ and shear module G_0.

The upper side $y = h$ of a bar is loaded with distributed tangential forces on the direction Oz, that induced an anti-plane deformation of the system layer-bar in the same direction.

The problem is to determine the tangential forces distribution $\tau_+(x)$ when in the contact zone $-a \le x \le a$ the pre-assigned regime of displacements is realized. The displacements are given as a smooth enough function $f(x)$, that is when $w_0(x,0) = f(x)(-a \le x \le a)$, where $w_0(x,y)$ is the only non-vanishing component of the bar displacements by Oz direction.

II. Here the elastic layer is replaced by a wedge-shaped base, which is presentable as r, ϑ, z in cylindrical coordinate system. The wedge with a shear module G is rigidly clamped. The load on the bar is given by forces with intensity $\tau_+(r)$ $\{a \le r \le b\}$, directed along Oz axis tangentially to its upper face (Fig.3).

The problem is to find a function $\tau_+(r)$ such that $w_0(r,\alpha) = f(r)$ $(a \le r \le b)$ where $f(r)$ is a given continuous function.

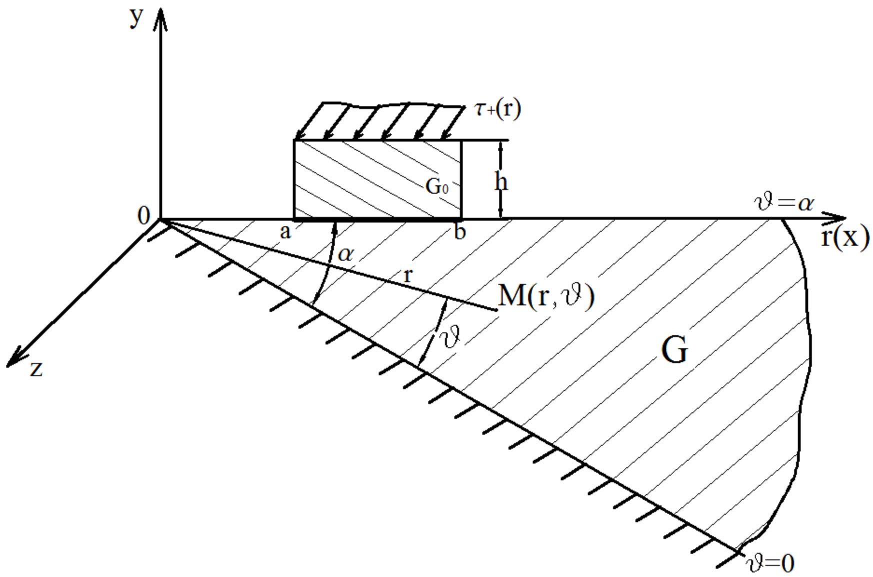

Fig.3

III. Here the elastic layer is replaced by a half-plane, and the analog of the case I is considered.

To deduce the main equations of posed problems, as a preliminary, let us construct solutions of some auxiliary boundary problems, that connect stresses with displacements in the contact zone.

For the case I we have

$$\begin{cases} \dfrac{\partial^2 w_0}{\partial x^2} + \dfrac{\partial^2 w_0}{\partial y^2} = 0 & (a < x < b;\ 0 < y < h) \\ \tau_{xz}\big|_{x=a} = G_0 \dfrac{\partial w_0}{\partial x}\bigg|_{x=a} = \tau_{xz}\big|_{x=b} = G_0 \dfrac{\partial w_0}{\partial x}\bigg|_{x=b} = 0 & (0 \le y \le h); \\ \tau_{yz}\big|_{y=0} = G_0 \dfrac{\partial w_0}{\partial y}\bigg|_{y=0} = \tau_-(x); \tau_{yz}\big|_{y=h} = G_0 \dfrac{\partial w_0}{\partial y}\bigg|_{y=h} = \tau_+(x) & (a \le x \le b) \end{cases} \tag{1.1}$$

where tangential stresses in contact zone is $\tau_-(x)$, and τ_{xz}, τ_{yz} are tangential components of stresses.

The b.v.p. (1.1) is considered with the following additional condition of the equilibrium of the bar or rectangle

$$\int_a^b \tau_-(x)\,dx = \int_a^b \tau_+(x)\,dx = T, \tag{1.2}$$

where T is a given quantity.

The solution of (1.1) can be build by means of finite Fourier cosine-transformation.

Similar to that in [12, (Ch III-13)] for Dirichlet class function $f(x)$, in the case under consideration one can derive

$$\overline{f}_c(n)=\int_a^b f(x)\cos\left[\mu_n(x)\right]dx;\quad \mu_n(x)=\pi n(x-a)/(b-a)\quad (n=0,1,2,...),$$

$$f(x)=\frac{\overline{f}_c(0)}{b-a}+\frac{2}{b-a}\sum_{n=1}^{\infty}\overline{f}_c(n)\cos\left[\mu_n(x)\right]\qquad (a<x<b). \tag{1.3}$$

Applying (1.3) to the b.v.p. (1.1)–(1.2) we arrive at the more compact form of its presentation

$$\begin{cases}\dfrac{d^2\overline{w}_0}{dy^2}-\dfrac{\pi n}{(b-a)^2}\ \overline{w}_0=0 & (0<y<h),\\ \left.\dfrac{d\overline{w}_0}{dy}\right|_{y=0}=\dfrac{\overline{\tau}_-(n)}{G_0};\ \left.\dfrac{d\overline{w}_0}{dy}\right|_{y=h}=\dfrac{\overline{\tau}_+(n)}{G_0} & (n=0,1,2,...),\end{cases} \tag{1.4}$$

where $\overline{w}_0=\overline{w}_0(n,y)$ and, in accordance with (1.3),

$$\left\{\overline{w}_0(n,y);\ \overline{\tau}_\pm(n)\right\}=\int_a^b\left\{w_0(x,y);\tau_\pm(x)\right\}\cos\left[\mu_n(x)\right]dx\quad (n=0,1,2,...).$$

The solution of (1.4) is

$$\overline{w}_0(n,y)=\begin{cases}\dfrac{1}{\lambda_n G_0\mathrm{sh}(\lambda_n h)}\left\{\overline{\tau}_+(n)\mathrm{ch}(\lambda_n y)-\overline{\tau}_-(n)\mathrm{ch}\left[\lambda_n(y-h)\right]\right\};\\ \qquad\qquad (0\le y\le h;\ \lambda_n=\pi n/(b-a);\ n=1,2,...);\\ \dfrac{\overline{\tau}(0)}{G_0}y+B_0\quad (0\le y\le h,\ n=0;\ \overline{\tau}(0)=\overline{\tau}_+(0)=\overline{\tau}_-(0)=T),\end{cases}$$

hence from (1.3) we get

$$w_0(n,y)=\frac{\overline{w}_0(0,y)}{b-a}+\frac{2}{b-a}\sum_{n=1}^{\infty}\overline{w}_0(n,y)\cos\left[\mu_n(x)\right]\quad (a\le x\le b,\ 0\le y\le h). \tag{1.5}$$

The analogous formulas take place for $\tau_\pm(x)$ as well.

Now from (1.5) it follows that

$$\overline{w}_0(x,0)=\frac{\overline{w}_0(0,0)}{b-a}+\frac{2}{(b-a)G_0}\sum_{n=1}^{\infty}\frac{\overline{\tau}_+(n)-\mathrm{ch}(\lambda_n h)\overline{\tau}_-(n)}{\lambda_n\mathrm{sh}(\lambda_n h)}\cos\left[\mu_n(x)\right] \tag{1.6}$$

$$\left(B_0=\overline{w}_0(0,0);\ a\le x\le b\right).$$

As it was noted above, the problem is considered under the following condition

$$w_0(x,0)=f(x)\quad (a<x<b),$$

where the function $f(x)$ is given. Represent it as the Fourier cosine series by the second formula in (1.3) and compare (1.3) with (1.6).

Then, for the basic three functions $\tau_{\pm}(x)$ and $f(x)$ of the problem under consideration, we arrive at the following relations between their Fourier cosine-coefficients

$$\begin{gathered}\lambda_n G_0 \operatorname{sh}(\lambda_n h)\,\bar{f}(n) = \bar{\tau}_+(n) - \operatorname{ch}(\lambda_n h)\,\bar{\tau}_-(n) \quad (n = 1, 2, \ldots) \\ \bar{w}_0(0,0) = \bar{f}(0); \quad \bar{w}_0(n,0) = \bar{f}(n); \ \lambda_n = \pi n/(b-a).\end{gathered} \tag{1.7}$$

For the thin rectangle $D_0\,(h << b-a)$ these relations are simplified.

Indeed, confining ourselves with terms of order h in power series of entire functions $\operatorname{sh}(\lambda_n h)$ and $\operatorname{ch}(\lambda_n h)$, formula (1.7) will take the following form

$$\lambda_n^2 G_0 h\, \bar{f}(n) = \bar{\tau}_+(n) - \bar{\tau}_-(n) \quad (n = 0, 1, 2, \ldots). \tag{1.8}$$

The relations obtained are discrete analogs of a thin bar-stringer deformation's differential equations in the well-known Melan model for a plane deformation (see [13, 14]). In the case of anti-plane deformation one has (see [15])

$$G_0 h \frac{d^2 w_0}{dx^2} = \tau_-(x) - \tau(x) \quad (a < x < b). \tag{1.9}$$

Now consider the case of an elastic layer, rigidly clamped by its side $y = -H$. In the base plane Oxy the corresponding b.v.p. for the layer $\Pi = \{-\infty < x < \infty;\ -H < y < 0\}$ is presented as

$$\begin{cases} \dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = 0 \quad ((x,y) \in \Pi) \\ \tau_{yz}\big|_{y=0} = G \dfrac{\partial w}{\partial y}\bigg|_{y=0} = \tau_-(x); \quad w\big|_{y=-H} = 0 \quad (-\infty < x < \infty); \end{cases} \tag{1.10}$$

where $w = w(x,y)$ is the only non-vanishing component of the bar displacements by Oz direction. The solution of (1.10) can be obtained by means of Fourier integral transform with respect to x. In the similar way, for boundary points displacements of a strip we have

$$w(x,0) = \frac{1}{\pi G} \int_{-a}^{a} \ln \operatorname{cth}\left(\frac{\pi |x-s|}{4H} \right) \tau_-(s)\, ds \quad (-\infty < x < \infty), \tag{1.11}$$

if the function $\tau_-(x)$ vanishes outside of the segment $-a \le x \le a$. The passage to the limit $H \to \infty$ yields

$$w(x,0) = \frac{1}{\pi G} \int_{-a}^{a} \ln \frac{1}{|x-s|}\, \tau_-(s)\, ds + C \quad (-\infty < x < \infty) \tag{1.12}$$

for the elastic half-plane $\Pi_- = \{-\infty < x < \infty; y < 0\}$.

Next, the corresponding b.v.p. for a wedge, clamped by its side $\vartheta = 0$ is

$$\begin{cases} \dfrac{\partial^2 w}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial w}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 w}{\partial \vartheta^2} = 0 \quad (0 < r < \infty;\ 0 < \vartheta < \alpha); \\ w\big|_{\vartheta=0} = 0;\ \ \tau_{\vartheta z}\big|_{\vartheta=\alpha} = G\,\dfrac{1}{r}\dfrac{\partial w}{\partial \vartheta}\bigg|_{\vartheta=\alpha} = \tau_-(r) \ \ (0 < r < \infty). \end{cases} \tag{1.13}$$

By means of Mellin integral transform with respect to r, its solution can be written as

$$w(r,\alpha) = \frac{1}{\pi G}\int_a^b \ln \frac{r^{\pi/2\alpha} + r_0^{\pi/2\alpha}}{\left|r^{\pi/2\alpha} - r_0^{\pi/2\alpha}\right|}\ \tau_-(r_0)\,dr_0 \quad (0 < r < \infty), \tag{1.14}$$

where the tangential stresses $\tau_-(x)$ vanishe outside of $[a,b]$ $(r = x > 0)$.

Now we are ready to present the main integral equations of posed contact problems.

On account of the character of problems considered, in formulas (1.11), (1.12) and (1.14) we are given displacements $w(x,0) = f(x)$.

From formula (1.11) the determination of tangential contact stresses $\tau_-(x)$ is reduced to the following Fredholm integral equations of the first kind

$$\frac{1}{\pi G}\int_{-a}^{a} \ln \operatorname{cth}\left(\frac{\pi|x-s|}{4H}\right)\tau_-(s)\,ds = f(x) \ \ (-a < x < a). \tag{1.15}$$

Its solution should satisfy the condition (1.2) with a and b replaced by $-a$ and a.

In terms of dimensionless coordinates and quantities, equation (1.15) can be written as

$$\frac{1}{\pi}\int_{-\alpha}^{\alpha} \ln \operatorname{cth}\left(\frac{|\xi-\eta|}{4}\right)\varphi(\eta)\,d\eta = g(\xi) \ \ (-\alpha < \xi < \alpha), \tag{1.16}$$

and condition (1.2)– as

$$\int_{-\alpha}^{\alpha} \varphi(\eta)\,d\eta = T_0 \quad (T_0 = \pi T/GH). \tag{1.17}$$

For a contact problem of an elastic half-plane, formulas corresponding to (1.12) take the form

$$\frac{1}{\pi G}\int_{-a}^{a} \ln\frac{1}{|x-s|}\tau_-(s)\,ds = f(x) + C \quad (-a < x < a),$$

and, if

$$\xi = x/a,\ \eta = s/a,\ \varphi(\xi) = \tau_-(a\xi)/G;\ g(\xi) = f(a\xi)/a$$

then

$$\frac{1}{\pi}\int_{-1}^{1} \ln\frac{1}{|\xi-\eta|}\varphi(\eta)\,d\eta = g(\xi) + C_0 \quad (-1 < \xi < 1). \tag{1.18}$$

The solution of (1.18) should satisfy the condition (1.17), where now $T_0 = T/aG$.
On the base of (1.14), a contact problem for a wedge is reduced to the integral equation

$$\frac{1}{\pi G}\int_a^b \ln\frac{r^{\pi/2\alpha}+r_0^{\pi/2\alpha}}{\left|r^{\pi/2\alpha}-r_0^{\pi/2\alpha}\right|}\,\tau_-\left(r_0\right)dr_0 = f\left(r\right) \qquad \left(a<r<b\right), \tag{1.19}$$

and putting

$$\xi=\left(r/a\right)^{\pi/2\alpha},\ \eta=\left(r_0/a\right)^{\pi/2\alpha};\quad \rho=\left(b/a\right)^{\pi/2\alpha};$$

$$\tau_0\left(\xi\right)=\frac{1}{G}\xi^{2\alpha/\pi-1}\tau_-\left(a\xi^{2\alpha/\pi}\right);\quad f_0\left(\xi\right)=\frac{\pi}{2\alpha a}f\left(a\xi^{2\alpha/\pi}\right),$$

we get

$$\frac{1}{\pi}\int_1^\rho \ln\frac{\xi+\eta}{\left|\xi-\eta\right|}\tau_0\left(\eta\right)d\eta = f_0\left(\xi\right) \qquad \left(1<\xi<\rho\right). \tag{1.20}$$

The condition (1.2) can be transformed to

$$\int_1^\rho \tau_0\left(\eta\right)d\eta = T_0 \qquad \left(T_0=\frac{\pi T}{2\alpha aG}\right). \tag{1.21}$$

Setting

$$\xi=\sqrt{\rho}\,e^{t/2},\ \eta=\sqrt{\rho}\,e^{u/2};\ \gamma=\ln\rho;\ -\gamma<t,\,u<\gamma;$$

$$\omega_0\left(t\right)=e^{t/2}\tau_0\left(\sqrt{\rho}\,e^{t/2}\right);\ g_0\left(t\right)=\frac{2}{\sqrt{\rho}}f_0\left(\sqrt{\rho}\,e^{t/2}\right)$$

in (1.20) we obtain the following integral equation with difference kernel

$$\frac{1}{\pi}\int_{-\gamma}^{\gamma}\ln\mathrm{cth}\left(\frac{\left|t-u\right|}{4}\right)\omega_0\left(u\right)du = g_0\left(t\right) \quad \left(-\gamma<t<\gamma\right),$$

which coincides with (1.16).

2. Solutions of main integral equations. The solutions of equations (1.16), (1.18), and (1.20) can be obtained with the use of the method, developed by M.G. Krein in works [9], [10, (Ch. IV-8)], dealing with a certain class of Fredholm integral equations of the second and first kind with symmetric difference kernels, closely connected to inverse problems of spectral theory of differential operators. Later on it was extended to more general classes of integral equations. There are lots of applied problems that can be described by integral equations with difference kernels. In monograph [18] and references therein one can find development of this theory.
The advantage of formulas derived by Krein is the absence there of Cauchy principal value improper integrals, and their quite an orderly analytic structure.
The main point of the method is that the solution of such an integral equation with an arbitrary continuous right hand side can be constructed by means of its solution with the right hand side identically equal to 1, if the last one exists and is unique.
Applying that to equation (1.16), present the desired solution as a sum of its symmetric and skew-symmetric parts

$\varphi(\xi)=\varphi_+(\xi)+\varphi_-(\xi);\; g(\xi)=g_+(\xi)+g_-(\xi);\; \varphi_\pm(-\xi)=\pm\varphi_\pm(\xi);\; g_\pm(-\xi)=\pm g_\pm(\xi),$
so consider

$$\frac{1}{\pi}\int_{-\alpha}^{\alpha}\ln\operatorname{cth}\left(\frac{|\xi-\eta|}{4}\right)\varphi_\pm(\eta)\,d\eta=g_\pm(\xi)\quad(-\alpha<\xi<\alpha). \tag{2.1}$$

Then, according to [9], [10], the unique integrable solution of

$$\frac{1}{\pi}\int_{-\alpha}^{\alpha}\ln\operatorname{cth}\left(\frac{|\xi-\eta|}{4}\right)q(\eta,\alpha)\,d\eta=1$$

is of the form

$$q(\xi,\alpha)=\left\{Q_{-1/2}(\operatorname{ch}\alpha)\sqrt{2(\operatorname{ch}\alpha-\operatorname{ch}\xi)}\right\}^{-1}\quad(-\alpha<\xi<\alpha),$$

where $Q_{-1/2}(\xi)$ is a Legendre function of the second kind.

The Krein function is

$$M(\alpha)=\int_0^{\alpha}q(\xi,\alpha)\,d\xi=\pi\,\mathrm{P}_{-1/2}(\operatorname{ch}\alpha)/2Q_{-1/2}(\operatorname{ch}\alpha).$$

Here $\mathrm{P}_{-1/2}(\xi)$ is a Legendre function of the first kind.

Relative to the argument $\xi=\operatorname{ch}\alpha$, the above functions can be presented by means of complete elliptic integrals of the first kind. With the use of formulas from [16, p.1036, f-las 8.851.1 and 8.851.2] one has

$$\mathrm{P}_{-1/2}(\operatorname{ch}\alpha)=\frac{2\sqrt{k}}{\pi}K(k'),\quad Q_{-1/2}(\operatorname{ch}\alpha)=2\sqrt{k}\,K(k)$$

$$k=e^{-\alpha};\; k'=\sqrt{1-k^2}=\sqrt{1-e^{-2\alpha}},$$

where $K(k)$ is a complete elliptic integral of the first kind of the modulus $k\,(0<k<1)$ and the complementary modulus k'.

Hence

$$M(\alpha)=\frac{1}{2}\,\frac{K(k')}{K(k)}=\frac{1}{2}\,\frac{K'}{K}\qquad(K'=K(k'))$$

and its derivative is

$$M'(\alpha)=\frac{1}{2K^2(k)}\left[K'(k')K(k)\frac{dk'}{d\alpha}-K'(k)K(k')\frac{dk}{d\alpha}\right].$$

The use of a differentiation formula for an elliptic integral relative to the modulus (see [16, p.921, f-la 8.123.2]) and the relation [16, p.921, f-la 8.122] yields

$$M'(\alpha)=\pi\left[4k'^2K^2(k)\right]^{-1}.$$

It now follows (see [9], [10]) that the even solution of (2.1) is

$$\varphi_+(\xi)=\left[\frac{1}{M'(\alpha)}\frac{d}{d\alpha}\int_0^{\alpha}q(\xi,\alpha)\;g_+(\xi)\,d\xi\right]q(\xi,\alpha)-$$

$$-\int_{\xi}^{\alpha}q(\xi,u)\frac{d}{du}\left[\frac{1}{M'(u)}\frac{d}{du}\int_0^{u}q(\eta,u)\;g_+(\eta)\,d\eta\right]du\quad(0<\xi<\alpha)$$

and its odd solution is

$$\varphi_-(\xi)=-\frac{d}{d\xi}\int_{\xi}^{\alpha}\frac{q(\xi,u)}{M'(u)}\left[\int_0^{u}q(\eta,u)\,dg_-(\eta)\right]du\quad(0<\xi<\alpha).$$

The inner integral here is understood in the sense of Stieltjes.

Note that equations (1.16), (2.1) appear also in mixed b.v.p. in the theory of a fluid stabilized filtration in strip shape porous grounds [17].

By the Krein method in [19] are presented solutions of i.e. (1.18) and some others with comparative analyses of various analytical methods.

Now, let us present solutions of (1.16) and (1.18) by means of spectral relationships, established in [11] via orthogonal functions method.

The solution of i.e. (1.16), (1.18) and (1.20) can be built by means of spectral relationships as follows

$$\int_1^{\rho}\ln\frac{\xi+\eta}{|\xi-\eta|}\frac{T_n(Y)\,d\eta}{\sqrt{(\rho^2-\eta^2)(\eta^2-1)}}=\lambda_n T_n(X)\quad(n=0,1,2,...;\quad 1<\xi<\rho)$$

$$Y=\cos\varphi,\;\;\varphi=\frac{\pi}{K'}\int_1^{\eta}\frac{du}{\sqrt{(u^2-1)(1-k^2u^2)}};\;k=1/\rho=(a/b)^{\pi/2\alpha};\tag{2.2}$$

$$X=\cos\vartheta,\;\;\vartheta=\frac{\pi}{K'}\int_1^{\xi}\frac{du}{\sqrt{(u^2-1)(1-k^2u^2)}};\;k'=\sqrt{1-k^2};\;\;K'=K(k');$$

$$\lambda_0=\pi K/\rho,\quad\lambda_n=\frac{1}{\rho n}K'\mathrm{th}(\pi nK/K')\qquad(n=1,2,...;\;K=K(k))$$

and the solution of i.e. (1.18) – as

$$\int_{-\alpha}^{\alpha}\ln\mathrm{cth}\left(\frac{|\xi-\eta|}{4}\right)\frac{T(V)\,d\eta}{\sqrt{2(\mathrm{ch}\alpha-\mathrm{ch}\eta)}}=\mu_n T_n(U)\quad(n=0,1,2,...;\quad-\alpha<\xi<\alpha)$$

$$\mu_n=2\lambda_n\,(n=1,2,...);\;\mu_0=2\lambda_0=2\pi\sqrt{k}\,K(k);\;k=e^{-\alpha};$$

$$U=\cos\Theta,\;\Theta=\frac{\pi}{2K'}e^{\alpha/2}\int_{-1}^{\xi}\frac{du}{\sqrt{2(\mathrm{ch}\alpha-\mathrm{ch}u)}};\;\;K'=K(k').\tag{2.3}$$

$$V=\cos\Phi,\ \Phi=\frac{\pi}{2K'}e^{\alpha/2}\int_{-\alpha}^{\eta}\frac{du}{\sqrt{2\left(\text{ch}\alpha-\text{ch}\,u\right)}};\ k'=\sqrt{1-e^{-2\alpha}};$$

Here $T_n\left(X\right)$ and $T_n\left(U\right)$ are Chebishev polynomials of the first kind. Their orthogonality conditions are of the form

$$\int_1^{\rho}T_n\left(X\right)T_m\left(X\right)\frac{d\xi}{\sqrt{\left(\rho^2-\xi^2\right)\left(\xi^2-1\right)}}=\begin{cases}K'/\rho & \left(m=n=0\right);\\ K'/2\rho & \left(m=n\neq 0\right);\\ 0 & \left(m\neq n\right);\end{cases}\qquad(2.4)$$

$$\int_{-\alpha}^{\alpha}T_n\left(U\right)T_m\left(U\right)\ \frac{d\xi}{\sqrt{2\left(\text{ch}\alpha-\text{ch}\xi\right)}}\ =\ \begin{cases}2e^{-\alpha/2}K' & \left(m=n=0\right);\\ e^{-\alpha/2}K' & \left(m=n\neq 0\right);\\ 0 & \left(m\neq n\right),\end{cases}\qquad(2.5)$$

where k and k' are taken from (2.2) and (2.3), respectively.

Integrals ϑ and Θ appearing there can be expressed as incomplete elliptic functions $F\left(\varphi,k\right)$ (see [16], p.260, f-la 3.152.9), namely

$$\int_1^{\xi}\frac{du}{\sqrt{\left(u^2-1\right)\left(1-k^2u^2\right)}}=\ F\left(\arcsin\left(\frac{\sqrt{\xi^2-1}}{\xi k'}\right),k'\right)\qquad\left(k'=\sqrt{1-1/\rho^2}\right);$$

$$\int_{-\alpha}^{\xi}\frac{du}{\sqrt{2\left(\text{ch}\alpha-\text{ch}u\right)}}=2e^{-\alpha/2}F\left(\arcsin e^{\frac{\alpha-\xi}{4}}\sqrt{\frac{\text{sh}\left[\left(\alpha+\xi\right)/2\right]}{\text{sh}\alpha}},k'\right)\quad\left(k'=\sqrt{1-e^{-2\alpha}}\right).$$

The solution of i.e. (1.20) we will find in the form of infinite series

$$\tau_0\left(\xi\right)=\frac{1}{\sqrt{\left(\rho^2-\xi^2\right)\left(\xi^2-1\right)}}\sum_{n=0}^{\infty}x_nT_n\left(X\right)\quad\left(1<\xi<\rho\right).\qquad(2.6)$$

For determining unknown coefficients x_n we substitute (2.6) in (1.20), interchange the order of summation and integration, use spectral relationships (2.2) and orthogonality conditions (2.4). As a result we obtain

$$x_0=\frac{\rho^2}{KK'}f_0^{(0)},\ x_n=\frac{2\pi\rho^2}{K'^2}n\,\text{cth}\left(\pi nK/K'\right)f_n^{(0)}\quad\left(n=1,2,...\right);$$
$$f_n^{(0)}=\int_1^{\rho}f_0\left(\xi\right)\frac{T_n\left(X\right)d\xi}{\sqrt{\left(\rho^2-\xi^2\right)\left(\xi^2-1\right)}}\qquad\left(n=0,1,2,...\right).\qquad(2.7)$$

Then, substituting (2.6) in (1.21) we get $T_0=x_0\,K'/\rho$, hence

$$T_0 = \rho f_0^{(0)} / K \tag{2.8}$$

in view of the first relation in (2.7).

On the other hand the function $f_0(\xi)$ can be considered as a sum

$$f_0(\xi) = \delta_0 + \tilde{f}_0(\xi) \quad \left(\tilde{f}_0'(\xi) \neq 0, \ 1 < \xi < \rho \right),$$

where δ_0 is the reduced rigid displacement of the rectangle D_0 in Oz direction. Note that $f_0(\xi) \equiv \delta \ \left(\tilde{f}_0(\xi) \equiv 0 \right)$ for the case of an absolutely rigid rectangle $(G_0 = \infty)$. Then

$$f_0^{(0)} = \delta_0 K'/\rho + \tilde{f}_0^{(0)}, \quad \tilde{f}_0^{(0)} = \int_1^\rho \frac{\tilde{f}_0(\xi)\, d\xi}{\sqrt{(\rho^2 - \xi^2)(\xi^2 - 1)}}$$

hence formula (2.8) establishes a certain connection of T_0 and δ_0.

Consider i.e. (1.16) and, as above, set

$$\varphi(\xi) = \frac{1}{\sqrt{2(\mathrm{ch}\alpha - \mathrm{ch}\xi)}} \sum_{n=0}^{\infty} y_n T_n(U) \quad (-\alpha < \xi < \alpha).$$

Repeating the same steps one can obtain

$$\begin{aligned} & y_0 = \frac{k g_0}{4\pi^2 K K'}, \quad y_n = \frac{nk}{2\pi K'^2} \mathrm{cth}\left(\pi n K/K'\right) g_n; \qquad (n = 1, 2, \ldots); \\ & g_n = \int_{-\alpha}^{\alpha} \frac{T_n(U)\, g(\xi)\, d\xi}{\sqrt{2(\mathrm{ch}\alpha - \mathrm{ch}\xi)}} \qquad (n = 0, 1, 2, \ldots), \end{aligned} \tag{2.9}$$

which leads to the connection of T_0 and g_0, taking into account condition (1.17) .

Finally, on the same way as above, consider i.e. (1.18). For

$$\varphi(\xi) = \frac{1}{\sqrt{1 - \xi^2}} \sum_{n=0}^{\infty} z_n T_n(\xi) \qquad (-1 < \xi < 1)$$

we make use of well-known spectral relationships (see [20, Ch. X])

$$\frac{1}{\pi} \int_{-1}^{1} \ln \frac{1}{|\xi - \eta|} \frac{T_n(\eta)\, d\eta}{\sqrt{1 - \eta^2}} = \begin{cases} \dfrac{1}{n} T_n(\xi) \quad (n = 1, 2, \ldots); \\ \qquad\qquad (-1 < \xi < 1); \\ \ln 2 \qquad\qquad (n = 0). \end{cases}$$

For the case under consideration we have $T_0 = T/aG$, in (1.17), which leads to

$$z_0 = T_0/\pi, \quad C_0 = (T_0 \ln 2 - g_0)\Big/\pi; \ z_n = \frac{2}{\pi} n g_n \qquad (n = 1, 2, \ldots)$$

$$g_n = \int_{-1}^{1} g(\xi) \frac{T_n(\xi)\, d\xi}{\sqrt{1-\xi^2}} \qquad (n = 0, 1, 2, \ldots). \tag{2.10}$$

Therefore,

$$\varphi(\xi) = \frac{1}{\sqrt{1-\xi^2}} \left(T_0 + 2\sum_{n=1}^{\infty} n g_n T_n(\xi) \right) \quad (-1 < \xi < 1). \tag{2.11}$$

It is not difficult to verify that the series (2.11) converges uniformly if $g(\xi)$ is continuously differentiable.

In conclusion of this section let us turn to equation (1.19) again. Putting there $a = 0$,

$$\xi = (r/b)^{\pi/2\alpha}, \ \eta = (r_0/b)^{\pi/2\alpha}; \ \tau_0(\xi) = \frac{1}{G} \xi^{2\alpha/\pi - 1} \tau_0\left(b\xi^{2\alpha/\pi}\right);$$

$$f_0(\xi) = \frac{\pi}{2\alpha a} f\left(ba\xi^{2\alpha/\pi}\right) \qquad (0 < \xi < 1),$$

we get

$$\frac{1}{\pi}\int_0^1 \ln \frac{\xi + \eta}{|\xi - \eta|} \tau_0(\eta)\, d\eta = f_0(\xi) \quad (0 < \xi < 1). \tag{2.12}$$

In [21, Ch. III-8] it is shown that the stresses for the wedge of angle α have order $r^{\pi/\alpha - 1}$, if displacements on its bounds are zero. Therefore the stresses at the wedge vertex have singularity, when $\alpha > \pi$. On the other hand

$$\tau_0(\xi) \sim \frac{1}{G}\left(\xi^{2\alpha/\pi}\right)^{\pi/\alpha - 1} \xi^{2\alpha/\pi - 1} = \frac{\xi}{G} = 0(\xi) \quad (\xi \to 0),$$

so $\tau_0(0) = 0$. Then the odd extension of equation onto the segment $[-1, 0]$ leads to (1.18). For this case in relation (1.17) it should be putted $T_0 = 0$.

3. The determination of the function $\tau_+(x)$. Depending on the problem under consideration, the function $\tau_+(x)$ can be determined with the help of given solutions of (1.16), (1.18), (1.20), and relations (1.7). Let us present it for each of three above mentioned contact problems with accordingly chosen dimensionless variables.

For the case of a layer we have

$$\tau_+(x) = \frac{\overline{\tau}_+(o)}{2a} + \frac{1}{a}\sum_{n=1}^{\infty} \overline{\tau}_+(n) \cos\left[\frac{\pi n (x+a)}{2a}\right] \quad (-a < x < a),$$

$$\overline{\tau}_+(n) = \int_{-a}^{a} \tau_+(x) \cos\left[\frac{\pi n (x+a)}{2a}\right] dx \qquad (n = 0, 1, 2, \ldots).$$

In accordance with (1.16) set

$$x = H\xi/\pi; \quad \chi_+(\xi) = \tau_+(H\xi/\pi)/G_0 \qquad (-\alpha < \xi < \alpha),$$

that is readily transformed to

$$\chi_+(\xi) = \frac{\chi_0^+}{2\alpha} + \frac{1}{\alpha}\sum_{n=1}^{\infty}\chi_n^+ \cos\left[\frac{\pi n(\xi+\alpha)}{2\alpha}\right] \quad (-\alpha < \xi < \alpha);$$

$$\chi_0^+ = T_0 \;\; (T_0 = \pi T/G_0 H); \;\; \chi_n^+ = H_0 \mathrm{sh}(\pi n h_0/2) n\bar{g}_n - \tag{3.1}$$

$$-k\mathrm{ch}(\pi n h_0/2)\varphi_n \quad (n = 1, 2, ...); \; k_0 = G/G_0; \;\; h_0 = h/a, \; H_0 = H/2a;$$

$$\left\{\chi_n^+, \varphi_n, \bar{g}_n\right\} = \int_{-\alpha}^{\alpha}\left\{\chi_+(\xi), \varphi(\xi), g(\xi)\right\}\cos\left[\frac{\pi n(\xi+\alpha)}{2\alpha}\right]d\xi \;\; (n = 0, 1, 2, ...).$$

Here $\varphi(\xi)$ and $g(\xi)$ are solution and the right hand side of equation (1.16) respectively, and formula for χ_n^+ is derived from (1.7). Now, in accordance with (1.8), the simplified form of χ_n^+ is

$$\chi_n^+ = \frac{1}{2}\pi n^2 h_0 H_0 \bar{g}_n - k_0\varphi_n \quad (n = 1, 2, ...).$$

For a thin rectangle $D_0(h << a)$ a function $\chi_+(\xi)$ can be determined from (1.9) with the use of the Melan model, namely

$$\chi_+(\xi) \approx k_0\varphi(\xi) - \frac{\pi h_0}{2H_0}g''(\xi) \qquad (-\alpha < \xi < \alpha). \tag{3.2}$$

To the case of a wedge it corresponds to i.e. (1.21) and analogs of relations (3.1) and (3.2) respectively are

$$\chi_+(\xi) = \tau_+\left(a\xi^{2\alpha/\pi}\right)/G_0 = h_+(\xi)\xi^{1-2\alpha/\pi} \qquad (1 < \xi < \rho)$$

$$h_+(\xi) = \frac{h_0^+}{\rho_0} + \frac{2}{\rho_0}\sum_{n=1}^{\infty}h_n^+ \cos\left[\frac{\pi n\left(\xi^{2\alpha/\pi} - 1\right)}{\rho_0}\right],$$

$$\rho_0 = \rho^{2\alpha/\pi} - 1; \;\; h_0^+ = T_0 \quad (T_0 = \pi T/2\alpha a G_0); \tag{3.3}$$

$$h_n^+ = k_0\mathrm{ch}(\pi n h_0/\rho_0)\tau_n^{(0)} + \frac{4\alpha^2}{\pi\rho_0}n\,\mathrm{sh}(\pi n h_0/\rho_0)\, f_n^{(0)} \quad (n = 1, 2, ...);$$

$$\left\{h_n^+, \tau_n^{(0)}, f_n^{(0)}\right\} = \int_1^{\rho}\left\{h_+(\xi), \tau_0(\xi), f_0(\xi)\right\}\cos\left[\frac{\pi n\left(\xi^{2\alpha/\pi} - 1\right)}{\rho_0}\right]d\xi;$$

$$\chi_+(\xi) \approx k_0 \xi^{1-2\alpha/\pi} \varphi(\xi) - \frac{\pi h_0}{2\alpha} \xi^{1-4\alpha/\pi} \left[\left(1 - \frac{2\alpha}{\pi}\right) f_0'(\xi) + \xi f_0''(\xi) \right].$$

For the case of half-plane we get

$$\chi_+(\xi) = \tau_+(a\xi) \Big/ G_0 = \frac{\chi_0^+}{2} + \sum_{n=1}^{\infty} \chi_n^+ \cos\left[\frac{\pi n(\xi+1)}{2}\right] \quad (-1 < \xi < 1);$$

$$\chi_0^+ = T_0 \;\; (T_0 = T/aG_0); \;\; \chi_n^+ = k_0 \mathrm{ch}(\pi n h_0/2)\varphi_n +$$

$$+\frac{1}{2}\pi n \,\mathrm{sh}(\pi n h_0/2)\bar{g}_n, \quad \chi_+(\xi) \approx k_0\varphi(\xi) - h_0 g''(\xi) \; (-1 < \xi < 1) \; (n = 1, 2, \ldots);$$

$$\left\{\chi_n^+, \varphi_n, \bar{g}_n\right\} = \int_{-1}^{1} \left\{\chi_+(\xi), \varphi(\xi), g(\xi)\right\} \cos\left[\frac{\pi n(\xi+1)}{2}\right] d\xi. \tag{3.4}$$

Note that there is a certain link between coefficients appearing in (3.1)–(3.4) and (2.7), (2.9) and (2.10). For the sake of derivations simplicity considering the case III, one has

$$\varphi_n = \int_{-1}^{1} \varphi(\xi) \cos\left[\frac{\pi n(\xi+1)}{2}\right] d\xi \quad (n = 0, 1, 2, \ldots).$$

Substituting here $\varphi(\xi)$ from (2.11) it is not difficult to obtain

$$\varphi_n = \frac{T_0}{\pi} I_{0n} + \frac{2}{\pi} \sum_{m=1}^{\infty} m g_m I_{mn}; \;\; I_{mn} = \int_0^{\pi} \cos\left[\frac{\pi n}{2}(\cos t + 1)\right] \cos(mt)\, dt$$

$$(m, n = 0, 1, 2, \ldots).$$

The function $g(\xi)$ can be approximated by linear combinations of Chebyshev polynomials of the first kind

$$g(\xi) \approx \frac{g_0}{\pi} + \frac{2}{\pi} \sum_{m=1}^{N} g_m T_m(\xi) \quad (-1 < \xi < 1).$$

Then

$$\varphi_n \approx \frac{T_0}{\pi} I_{0n} + \frac{2}{\pi} \sum_{m=1}^{N} m g_m I_{mn}$$

The obtained integrals I_{mn} are Fourier cosine-coefficients, and can be calculated by the method of least squares (see [22, Ch. IV-11]) up to required precision.

Conclusion

In the paper a new formulations of contact problems are suggested. For two elastic solids, fastened to each other by some part of their surfaces, the effect of pre-assigned regime of displacements on the contact surface is studied. In such a setting contact problems for solids of three different configurations under anti-plane deformation are solved. These problems are reduced to the Fredholm integral equations of the first kind and their solutions are built in complete form by both the Krein method and the method of orthogonal functions. The approach presented here can be efficiently applied also to a plane and axially symmetric contact problems.

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