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**GREEN'S FUNCTION APPROACH IN APPROXIMATE
CONTROLLABILITY PROBLEMS**

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Ключевые слова: приближенная управляемость, система с распределенными параметрами, функция Грина, бесконечная струна, полубесконечный стержень, источник с кусочно-постоянной интенсивностью, оптимизация параметра

Բանալի բառեր. գրեթե ղեկավարելիություն, բաշխված պարամետրերով համակարգ, Գրինի ֆունկցիա, անվերջ լար, կիսաանվերջ ձող, կրոր առ կրոր հասարակուն ինֆինիտիվությամբ աղբյուր, պարամետրի օպտիմալացում

Метод функции Грина в задачах о приближенной управляемости

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В настоящей статье предлагается математический метод решения задач управления, основанный на использовании функции Грина. Записав функцию состояния с помощью формулы Грина и подставив в требуемые условия, находятся управляющие функции, обеспечивающие приближенную управляемость исследуемой системы, в явном виде. Выбрав управляющую функцию соответствующим образом, требуемые условия обеспечиваются с необходимой точностью.

Приводятся примеры, иллюстрирующие определение управляющих функций. В частности, рассматриваются бесконечная струна, управляемая сосредоточенной силой, полубесконечный стержень, нагреваемый точечным источником тепла, конечный стержень, нагреваемый с границы и оптимизация параметров электрической цепи. Обсуждаются основные результаты вычислений.

Գրինի ֆունկցիայի եղանակը գրեթե ղեկավարելիության խնդիրներում

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Սույն աշխատանքում առաջարկվում է ղեկավարման խնդիրների լուծման մաթեմատիկական մոդելում հիմնված Գրինի ֆունկցիայի եղանակի վրա: Վիճակի հավասարման լուծումը ներկայացնելով Գրինի բանաձևով և փոխարինելով պահանջվող վերջնական պայմանների մեջ՝ կառուցվում են ուսումնասիրվող համակարգի գրեթե ղեկավարելիությունն ապահովող ղեկավարումներ՝ բացահայտ րեսքով: Ընտրելով ղեկավարման ֆունկցիան համապարասխան կերպ՝ պահանջվող վերջնական պայմանները բավարարվում են բավարար ճշգրտությամբ:

Բերվում են ղեկավարումների որոշումը պարզաբանող օրինակներ: Մասնավորապես դիտարկվում են կեպային ուժով ղեկավարվող անվերջ լարի, կեպային աղբյուրով փաթացվող կիսաանվերջ ձողի և ծայրից

տաքացվող վերջավոր հեծանի ղեկավարման և էլեկտրական շղթայի բնութագրիչների օպտիմալացման խնդիրներ: Քննարկվում են թվային հաշվարկի հիմնական արդյունքները:

A mathematical approach based on Green's function approach allowing to construct controls providing approximate controllability is suggested in the present paper. Representing the solution of governing system via Green's formula and substituting it in prescribed terminal conditions, we obtain control functions providing approximate controllability of the system under study in explicit form. Choosing appropriate controls, we can provide required accuracy of approximation for prescribed conditions.

Examples illustrating the procedure are described. Particularly, infinite string, controlled by a concentrated force, semi-infinite rod heated by a point heat source, finite rod heated from its boundary and parameter optimization for electrical circuit are considered. Results of computations are brought.

Introduction

Control systems are investigated in theory in order to be implemented in practice. Before being implemented, system's efficiency, expenses, requirements, reliability, areas and term of use, etc. are analyzed. During implementation we need not only a guarantee that it is possible to control the system, i.e. for *controllability*, but also the corresponding controls in explicit form. Nowadays efficient numerical techniques allow to approximate mathematical models of even very complex control systems, therefore theoretically it is much more important to analyze the controllability.

In many types of control problems, depending on state equation, type of control (boundary or distributed) or required terminal state, it is becoming impossible to ensure *exact* controllability, i.e. exact implementation of given conditions in given time, even if at our disposal we have controls from quite wide classes of functions. For instance, the main part of nonlinear control systems, even some linear control systems, systems defined in unbounded domains, etc., are not exactly controllable. Sometimes, by a specific choice of control parameter, prescribed terminal conditions may be implemented approximately:

$$\|w_p - w_i\| \leq \varepsilon,$$

where w_p is the prescribed, and w_i is the implemented states, ε is a given positive number, the norm is understood in a reasonable sense.

This concept may arise, for instance, when by a choice of controls the energy of a particular system sufficiently decreases but does not equal to zero exactly.

Particularly, in [1] a class of controls ensuring the approximate controllability (the system is not exact controllable) of a rectangular elastic plate lying on a Winkler base with controllable distribution function subjected to uniformly moving load is explicitly represented. A numerical scheme based on the Bubnov–Galerkin procedure is suggested and the approximate controllability is derived in particular cases.

The approximate controllability of semilinear heat equation with some types of nonlinearities is considered in [2]. It is proved that if the control acts on a bounded subset of the domain, the system is approximate controllable, but not exact controllable. For further detailed review see [3].

There are several numerical approaches to analyze a given system to exact or approximate controllability. For systematic report of those approaches we refer to [3]. Even though those approaches are universal enough, there are some problems on controllability unsolved yet [4].

Approximate controllability may play an important role in long- or infinite-time control problems. In some processes in industry, finance, engineering, we encounter with necessity of achieving required state for the system and holding on that regime for practically a long time. This problem can be viewed as a problem of determination of controls transferring its initial state into a sufficiently narrow neighborhood of prescribed state (resolving approximate controllability problem) and at the same time ensuring that the state will stay in that neighborhood with $t \rightarrow \infty$ (ensuring stability). This article is intended to establish solution to the first part of the problem.

In this order here we represent the solution of the governing system via Green's function, and then, substituting it into prescribed terminal conditions, derive implicit representation for admissible controls. Due to non-uniqueness of solution, we chose the parameters of the control in such a way that the terminal conditions are satisfied with required accuracy. The approach is especially useful if Green's function of the system under study is known or its construction is easier than the study of the system.

We have chosen Green's function approach because it is very convenient in cases when the analysis of the governing boundary value problem is very complicated, but its Green's function is known from handbooks or other resources. Green's function approach is widely used in deformable body mechanics. In handbook [5], Green's functions for many ordinary and partial differential equations and their coupled systems, differential-difference equations, integral and integro-differential equations are brought explicitly. Handbooks [6–8] contain exact solutions of general equations and coupled systems of equations in terms of Green's function and not only: the procedure suggested in this article allows to solve control problems also for systems whose explicit solution is defined not necessarily by Green's function.

Note, that Green's function approach was used to solve control problems earlier, see, for instance, [9–11] and references therein.

1 Green's function approach and implicit representation of control functions

Suppose we deal with a mechanical system, the state of which is described by

$$\mathcal{D}[w] = f(\mathbf{x}, t, u_d) \quad \text{in } (\mathbf{x}, t) \in \Omega \times (0, T) := \mathcal{O}, \quad (1.1)$$

subject to boundary condition

$$\mathcal{B}[w] = u_b(t) \quad \text{in } (\mathbf{x}, t) \in \partial\Omega \times [0, T]. \quad (1.2)$$

Assume that the initial state of the system is given:

$$\mathcal{I}[w] = 0 \quad \text{at } t = 0, \quad \mathbf{x} \in \bar{\Omega}. \quad (1.3)$$

Above $\mathcal{D}[\cdot]$, $\mathcal{B}[\cdot]$ and $\mathcal{I}[\cdot]$ are linear operators of state, boundary and initial conditions, $f : \mathcal{O} \times \mathcal{U}_d \rightarrow \mathbb{R}$ is given right-hand side, $\Omega \subseteq \mathbb{R}^3$ is the finite or infinite domain of the system, in the case when it is finite $\partial\Omega$ is its boundary and $\bar{\Omega}$ – the ordinary closure, time moment T is fixed and given. We assume that Green's function for those inputs is known or may be constructed. The control can be carried out either via u_d (distributed control) or u_b (boundary control). Our aim is to implement such controls (distributed or boundary) that some required conditions are satisfied at T :

$$\mathcal{T}[w] = 0 \quad \text{at } t = T, \quad \mathbf{x} \in \Omega. \quad (1.4)$$

The sets of such controls we denote by \mathcal{U}_d and \mathcal{U}_b , respectively, and call sets of admissible controls. If the set of admissible (distributed or boundary) controls is non-empty, system (1.1)–(1.3) is called controllable (in corresponding sense). If for some control function (1.4) holds exactly, system (1.1)–(1.3) is called exact controllable, and if $\|\mathcal{T}[w]\| \leq \varepsilon$ uniformly in Ω with sufficiently small $\varepsilon > 0$, it is called approximate controllable. Norm $\|\cdot\|$ is understood in a reasonable sense.

Denote Green's function of (1.1)–(1.3) by $G(\mathbf{x}, \boldsymbol{\xi}, t, \tau)$: it is the solution of the initial–boundary value problem

$$\mathcal{D}^{(\mathbf{x}, t)} [G(\mathbf{x}, \boldsymbol{\xi}, t, \tau)] = \delta(\mathbf{x} - \boldsymbol{\xi}) \delta(t - \tau) \quad \text{in } \mathbf{x}, t, \boldsymbol{\xi}, \tau \in \bar{\mathcal{O}}, \quad (1.5)$$

$$\mathcal{B}_0^{(\mathbf{x}, t)} [G(\mathbf{x}, \boldsymbol{\xi}, t, \tau)] = 0 \quad \text{in } (\mathbf{x}, t) \in \partial\Omega \times [0, T]. \quad (1.6)$$

$$\mathcal{I}_0^{(\mathbf{x}, t)} [G(\mathbf{x}, \boldsymbol{\xi}, t, \tau)] = 0 \quad \text{at } t = 0, \quad \mathbf{x} \in \bar{\Omega}. \quad (1.7)$$

Here the superscript (\mathbf{x}, t) means that the corresponding operator acts on the indicated variables, and subscript 0 corresponds to homogeneous parts, $\delta(\mathbf{x})$ denotes Dirac's spatial, and $\delta(t)$ – one dimensional functions. If the solution of (1.5)–(1.7) is known, the solution of (1.1)–(1.3) is defined through the expression

$$w(\mathbf{x}, t) = \int_0^t \int_{\bar{\Omega}} G(\mathbf{x}, \boldsymbol{\xi}, t, \tau) W(\boldsymbol{\xi}, \tau, u_d, u_b) d\boldsymbol{\xi} d\tau, \quad (1.8)$$

in which $W(\boldsymbol{\xi}, \tau, u_d, u_b)$ is a distribution depending on f , and u_b linearly [5].

Forcing (1.8) to satisfy (1.4) at $t = T$, we derive implicit representation for unknown function:

$$\mathcal{T} \left[\int_0^t \int_{\Omega} G(\mathbf{x}, \boldsymbol{\xi}, t, \tau) W(\boldsymbol{\xi}, \tau, u_d, u_b) d\boldsymbol{\xi} d\tau \right] = 0, \quad (1.9)$$

whether it be distributed or boundary control function.

However, in practice, implicit representation of control function does not contain enough information for control system implementation and one needs their explicit representation. Since, in general, (1.9) is a nonlinear constraint on control function, therefore very often (1.9) cannot be satisfied exactly by any choice of admissible control (boundary or distributed). In such cases we will search for controls providing approximate satisfaction of (1.9) (controllability of (1.1)–(1.3)) with sufficient accuracy.

Suppose $\mathcal{T}[w(\mathbf{x}, t)] = w(\mathbf{x}, T) - w_T(\mathbf{x})$, then

$$\begin{aligned} \int_0^T \int_{\Omega} G(\mathbf{x}, \boldsymbol{\xi}, T, \tau) f(\boldsymbol{\xi}, \tau, u_d) d\boldsymbol{\xi} d\tau &= w_T(\mathbf{x}) - \\ &- \int_0^T \int_{\Omega} G(\mathbf{x}, \boldsymbol{\xi}, T, \tau) W_0(\boldsymbol{\xi}, \tau, u_b) d\boldsymbol{\xi} d\tau, \end{aligned}$$

a.e. in Ω , in which

$$W(\boldsymbol{\xi}, \tau, u_d, u_b) = W_0(\boldsymbol{\xi}, \tau, u_b) + f(\boldsymbol{\xi}, \tau, u_d).$$

In particular, if f is linear in u_d , $f(\boldsymbol{\xi}, \tau, u_d) = f_0(\boldsymbol{\xi}, \tau) + u_d(\tau)$ for distributed controls we have

$$\begin{aligned} \int_0^T \int_{\Omega} G(\mathbf{x}, \boldsymbol{\xi}, T, \tau) u_d(\tau) d\boldsymbol{\xi} d\tau &= w_T(\mathbf{x}) - \int_0^T \int_{\Omega} G(\mathbf{x}, \boldsymbol{\xi}, T, \tau) f_0(\boldsymbol{\xi}, \tau) d\boldsymbol{\xi} d\tau - \\ &- \int_0^T \int_{\Omega} G(\mathbf{x}, \boldsymbol{\xi}, T, \tau) W_0(\boldsymbol{\xi}, \tau, u_b) d\boldsymbol{\xi} d\tau, \end{aligned}$$

a.e. in Ω .

Similar expressions one can obtain also for boundary controls u_b . Right hand sides of derived equations are given, while left hand sides depend on controls that must be determined. Beside derived equalities, control function must satisfy admission conditions of (1.1)–(1.4). Searching unknown function in specific forms (power, trigonometric or other orthogonal polynomials, piecewise functions, etc.), one can make both sides of derived equalities close enough. In case of presence of a free parameter in control function, it is possible to consider the minimization problem

$$\|w(\mathbf{x}, T) - w_T(\mathbf{x})\| \rightarrow \min$$

with reasonable norm.

2 Examples and explicit representation of controls

Now let us demonstrate control function determination procedure in some particular cases, in which Green's function approach is interesting to compare with other usual approaches.

Example 1. Control of infinite string vibrations via concentrated controls

Let us consider infinitely long string controlled by a concentrated force with controllable time-dependent intensity. The governing equation for displacement of the string reads as

$$\frac{\partial^2 w}{\partial t^2} - c^2 \frac{\partial^2 w}{\partial x^2} = f(x, t, u), \quad (x, t) \in \mathbb{R} \times (0, T), \quad (2.1)$$

$$f(x, t, u) = \mathcal{A}_{[0, T]}[u] \delta(x - x_0), \quad x_0 \in \mathbb{R},$$

in which c is the velocity of wave propagation in the string, u is the control impact, in general the operator $\mathcal{A}_{[0, T]}[u] = u(t) [\mathbf{1}(t) - \mathbf{1}(t - T)]$ is defined like in [1, 12–15], $\mathbf{1}(t)$ is the unit step function, x_0 is some finite point of the string. Assume the initial state of the string

$$w(x, 0) = w_0(x), \quad \left. \frac{\partial w}{\partial t} \right|_{t=0} = w_0^1(x), \quad x \in \mathbb{R}, \quad (2.2)$$

is known. Our aim is to find such bounded distributed controls u , $|u| \leq u_0$, that ensure the terminal conditions

$$w(x, T) = w_T(x), \quad \left. \frac{\partial w}{\partial t} \right|_{t=T} = w_T^1(x), \quad x \in \mathbb{R}, \quad (2.3)$$

with required accuracy ε .

The solution of (2.1)–(2.3) is given by (c.f. (1.8)) [5]

$$\begin{aligned} w(x, t) &= \\ &= \int_0^t \int_{-\infty}^{\infty} G(x - \xi, t - \tau) [f(\xi, \tau, u) + w_0(\xi) \delta'(\tau) + w_0^1(\xi) \delta(\tau)] d\xi d\tau := \\ &:= \Phi(x, t, u) + \Psi(x, t), \end{aligned} \quad (2.4)$$

in which δ' is understood in the sense of distributions,

$$\begin{aligned} G(x, t) &= \mathbf{1}(ct - |x|), \quad \Phi(x, t, u) = \frac{1}{2c} \int_0^t \mathbf{1}(c(t - \tau) - |x - x_0|) u(\tau) d\tau, \\ \Psi(x, t) &= \frac{1}{2c} \int_0^t \int_{-\infty}^{\infty} G(x - \xi, t - \tau) [w_0(\xi) \delta'(\tau) + w_0^1(\xi) \delta(\tau)] d\xi d\tau. \end{aligned}$$

It is easy to see that the solution decays when $x \rightarrow \pm\infty$. Thus, required controls must satisfy (c.f. (2.3))

$$\Phi(x, T, u) = w_T(x) - \Psi(x, T), \quad \left. \frac{\partial \Phi}{\partial t} \right|_{t=T} = w_T^1(x) - \left. \frac{\partial \Psi}{\partial t} \right|_{t=T}, \quad (2.5)$$

for almost all $x \in \mathbb{R}$. Here

$$\begin{aligned} \Psi(x, T) &= -\theta(0) \int_{-\infty}^{\infty} \left. \frac{\partial G(x - \xi, T - \tau)}{\partial \tau} \right|_{\tau=0} w_0(\xi) d\xi + \\ &\quad + \theta(0) \int_{-\infty}^{\infty} G(x - \xi, T) w_0^1(\xi) d\xi, \\ \frac{\partial \Phi}{\partial t} &= \frac{1}{2c} \int_0^t \delta(c(t - \tau) - |x - a|) u(\tau) d\tau, \end{aligned}$$

$$\frac{\partial \Psi}{\partial t} = \frac{1}{2c} \int_0^t \int_{-\infty}^{\infty} \delta(c(t-\tau) - |x-\xi|) [w_0(\xi) \delta'(\tau) + w_0^1(\xi) \delta(\tau)] d\xi d\tau.$$

We have taken into account that in the sense of distributions $\mathbf{1}' = \delta$. Above $\theta(0)$ is the Heaviside function when $t = 0$.

On the left-hand sides of (2.5) we have linear functionals on unknown $u \in \mathcal{U}$, while in the right-hand sides– the values of those functionals on particular admissible controls. Our aim is to find those particular functions. Since x is defined in unbounded domain, we will search for such a u that

$$\max_{x \in \mathbb{R}} |\Phi(x, T, u) + \Psi(x, T) - w_T(x)| \quad \text{and} \quad \max_{x \in \mathbb{R}} \left| \frac{\partial \Phi}{\partial t} \Big|_{t=T} + \frac{\partial \Psi}{\partial t} \Big|_{t=T} - w_T^1(x) \right| \quad (2.6)$$

uniformly converges to 0 with $x \rightarrow \pm\infty$.

Since in $[0, T]$ the point $x = x_0$ vibrates as u forces, we have

$$u(0) = \nu w_0(x_0) \quad \text{and} \quad u(T) = \nu w_T(x_0), \quad (2.7)$$

up to normalizing coefficient ν which will be specified, for instance, by introducing dimensionless quantities.

Thus, for u we have to solve boundary value problem (2.5), (2.7). From (2.5) at $x = x_0$ we have

$$\begin{aligned} \frac{1}{2c} \int_0^T u(\tau) \mathbf{1}(T-\tau) d\tau &= w_T(x_0) - \Psi(x_0, T), \\ \frac{\theta(0)}{2c^2} u(T) &= w_T^1(x_0) - \frac{\partial \Psi}{\partial t} \Big|_{x=x_0, t=T}. \end{aligned}$$

Here we have taken into account that

$$\delta(c(T-\tau)) = \frac{1}{c} \delta(T-\tau), \quad \mathbf{1}(c(T-\tau)) = \mathbf{1}(T-\tau).$$

By virtue of (2.7), the second restriction is a kind of consistency requirement for initial and terminal data:

$$\frac{\theta(0)}{2c^2} w_T(x_0) = w_T^1(x_0) - \frac{\partial \Psi}{\partial t} \Big|_{x=x_0, t=T}.$$

Thus, we have to solve the integral equation

$$\int_0^T u(\tau) \mathbf{1}(T-\tau) d\tau = \mathcal{M}, \quad \mathcal{M} = 2c [w_T(x_0) - \Psi(x_0, T)], \quad (2.8)$$

subjected to boundary conditions (2.7). Naturally, it is non-unique. For instance,

$$u(t) = \nu w_0(x_0) + \nu (w_T(x_0) - w_0(x_0)) t + u_1 t (T-t), \quad (2.9)$$

in which $u_1 = \text{const}$, satisfies (2.7), (2.8) with

$$u_1 = \frac{6}{T^3} \left[\mathcal{M} - \nu w_0(x_0) T - \nu (w_T(x_0) - w_0(x_0)) \frac{T^2}{2} \right].$$

Substituting (2.9) into (2.5) and choosing x_0 in a proper way, we can make both (2.6) sufficiently small. We also have to take into account that $|u| \leq u_0$.

Among continuous solutions trigonometric (or whatever else) polynomials also can be considered. Indeed, if we seek such a_k , b_k and ω_k that

$$u(t) = \sum_{k=1}^n [a_k \sin(\omega_k t) + b_k \cos(\omega_k t)], \quad (2.10)$$

then we have

$$\sum_{k=1}^n \frac{1}{\omega_k} [a_k (1 - \cos(\omega_k T)) + b_k \sin(\omega_k T)] = \mathcal{M}, \quad (2.11)$$

$$\sum_{k=1}^n b_k = \nu w_0(x_0), \quad \sum_{k=1}^n [a_k \sin(\omega_k T) + b_k \cos(\omega_k T)] = \nu w_T(x_0),$$

and can require to minimize some cost functional under these constraints [16].

In finite domains usually Fourier method of variables separation is used, providing efficient numerical scheme. In unbounded domains integral transforms are used, in some cases providing explicit solutions in integral form. Sometimes, d'Alembert's formula is used to derive explicit solution in finite domain [19]. We will compare now the implicit representations provided by Green's function approach and d'Alembert's formula. The general solution of (2.1), (2.2) according to d'Alembert formula is given by

$$\begin{aligned} w(x, t) &= \frac{w_0(x - ct) + w_0(x + ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} w_0^1(\xi) d\xi + \\ &+ \frac{1}{2c} \int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} \mathcal{A}_{[0,T]}[u](\tau) \delta(\xi - x_0) d\xi d\tau. \end{aligned}$$

Comparing this expression with (2.4) we see that, indeed,

$$\Phi(x, t, u) = \frac{1}{2c} \int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} \mathcal{A}_{[0,T]}[u](\tau) \delta(\xi - x_0) d\xi d\tau.$$

$$\Psi(x, t) = \frac{w_0(x - ct) + w_0(x + ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} w_0^1(\xi) d\xi,$$

i.e. both approaches provide the same implicit representation for control functions.

Example 2. Heating of semi-infinite rod

Suppose sufficiently thin semi-infinite rod is heated via point source, i.e. its temperature obeys heat equation on semi-axis:

$$\frac{\partial \Theta}{\partial t} - \alpha^2 \frac{\partial^2 \Theta}{\partial x^2} = f(x, t) \quad \text{in } (x, t) \in \mathbb{R}^+ \times (0, T), \quad (2.12)$$

$$f(x, t) = u(t) \delta(x - x_0), \quad x_0 \in \mathbb{R}^+,$$

subjected to boundary condition

$$\frac{\partial \Theta}{\partial x} = 0 \quad \text{in } x = 0, \quad t \in (0, T), \quad (2.13)$$

where x_0 is fixed, α^2 is the thermal diffusivity of the rod material. The initial temperature distribution in the rod is known:

$$\Theta(x, 0) = \Theta_0(x) \quad \text{in } x \in \mathbb{R}^+. \quad (2.14)$$

Our aim is the explicit representation of heating regimes u , $|u| \leq u_0$, implementing the terminal condition

$$\Theta(x, T) = \Theta_T(x) \quad \text{in } x \in \mathbb{R}^+$$

with required accuracy ε .

The solution of (2.12)–(2.14) is

$$\begin{aligned} \Theta(x, t) &= \int_0^t \int_0^\infty G(x, \xi, t - \tau) [f(\xi, \tau) + \Theta_0(\xi) \delta(\tau)] d\xi d\tau = \\ &= \int_0^t G(x, x_0, t - \tau) u(\tau) d\tau + \theta(0) \int_0^\infty G(x, \xi, t) \Theta_0(\xi) d\xi := \\ &:= \Phi(x, t, u) + \Psi(x, t), \end{aligned}$$

in which [5]

$$G(x, \xi, t) = \frac{1}{\sqrt{\pi}} \cdot \frac{1}{\sqrt{4\alpha^2 t}} \left\{ \exp \left[-\frac{(x - \xi)^2}{4\alpha^2 t} \right] + \exp \left[-\frac{(x + \xi)^2}{4\alpha^2 t} \right] \right\}.$$

It is easy to verify that Θ is uniformly bounded in $t \in [0, T]$ when $x \rightarrow \infty$.

Since the rod is sufficiently thin, we can always assume that all points of its cross section at every fixed moment have the same temperature, and its $x = x_0$ point is heated by u , then $\nu_0 \Theta(x_0, t) = u(t)$, $t \in [0, T]$, particularly, $u(0) = \nu_1 \Theta_0(x_0)$, $u(T) = \nu_2 \Theta_T(x_0)$, or

$$\Phi(x_0, T, u) + \Psi(x_0, T) = \Theta_T(x_0). \quad (2.15)$$

Here ν_0 , ν_1 and ν_2 play the same role as in previous example.

Repeating the procedure above, we will finally arrive at the implicit representation

for unknown controls

$$\int_0^T K(T-\tau)u(\tau)d\tau = \mathcal{M}, \quad \mathcal{M} = \Theta_T(x_0) - \Psi(x_0, T), \quad (2.16)$$

in which

$$K(t) := G(x_0, x_0, t) = \frac{1}{2\sqrt{\pi}} \cdot \frac{1}{\sqrt{\alpha^2 t}} \left\{ 1 + \exp\left[-\frac{x_0^2}{\alpha^2 t}\right] \right\}.$$

Besides solutions indicated in the last example, in the case of (2.16) we can also use the form

$$u(t) = \frac{\mathcal{M}}{K(T-t)}L(t),$$

in which L ensures the inclusion $u \in L^2[0, T]$ (say) and satisfies

$$L(0) = \nu \frac{K(T)}{\mathcal{M}} \Theta_0(x_0), \quad \lim_{t \rightarrow T} \frac{L(t)}{K(T-t)} = \nu \frac{1}{\mathcal{M}} \Theta_T(x_0), \quad \int_0^T L(t) dt = 1.$$

It is a usual procedure to approximate the solution of (2.16) by families of orthogonal functions (Bessel functions, Chebyshev (or whatever else) polynomials, etc.). In any case we have to take into account the restriction $|u| \leq u_0$.

A physically reasonable solution may be derived if the set of admissible controls is $\mathcal{U}_e = \{u \in L^1[0, T]; |u| \leq u_0, \text{supp } u \subseteq [0, T]\}$. Then the unknown controls can be expressed as piecewise constant function [1, 14, 18]

$$u(t) = \sum_{k=1}^n u_k \theta(t - t_k), \quad 0 = t_0 < t_1 < \dots < t_{n-1} < t_n = T,$$

$$u_1 = \nu \Theta_0(x_0), \quad \sum_{k=2}^n u_k = \nu \Theta_T(x_0) - \nu \Theta_0(x_0) \quad \sum_{k=1}^n |u_k| \leq u_0, .$$

It characterizes switching intensity: during the whole interval (t_{k-1}, t_k) the source heats with temperature u_k . After (somehow) explicit representation of controls from (2.16) we face with providing the equality

$$\Phi(x, T, u) + \Psi(x, T) = \Theta_T(x), \quad (2.17)$$

for almost all $x \in \mathbb{R}^+$. It is very useful to mention that the point x_0 can serve as robustness parameter regulating the difference between required and implemented states (e.g. (2.6)).

Example 3. Boundary heating of finite rod

Let finite, sufficiently thin non-homogeneous rod is thermo-isolated from external medium and is heated from its boundary. The temperature of the rod obeys one-

dimensional heat equation

$$\frac{1}{\alpha^2} \frac{\partial \Theta}{\partial t} - (x^2 + a^2)^2 \frac{\partial^2 \Theta}{\partial x^2} = 0 \quad \text{in } (x, t) \in (0, l) \times (0, T), \quad (2.18)$$

where $\alpha^2 (x^2 + a^2)^2$ is the coordinate-dependent thermal diffusivity, subjected to boundary conditions

$$\Theta(0, t) = u(t), \quad \Theta(l, t) = 0, \quad t \in [0, T]. \quad (2.19)$$

The initial state of the rod

$$\Theta(x, 0) = \Theta_0(x), \quad x \in [0, l], \quad (2.20)$$

is known and our aim is to design boundary heating regime u , $|u| \leq u_0$, ensuring exact or approximate satisfaction of terminal state

$$\Theta(x, T) = 0, \quad x \in [0, l].$$

From admission conditions we have

$$u(0) = \nu \Theta_0(0), \quad \Theta_0(l) = 0, \quad u(T) = 0,$$

up to a constant ν standing for correspondence in dimensions.

The solution of (2.18)–(2.20) is

$$\begin{aligned} \Theta(x, t) &= \int_0^t \int_0^l G(x, \xi, t - \tau) [\Theta_0(\xi) \delta(\tau) - u(\tau)(\xi^2 + a^2)^2 \delta'(\xi)] d\xi d\tau = \\ &= \theta(0) \int_0^l G(x, \xi, t) \Theta_0(\xi) d\xi - \int_0^t u(\tau) G_0(x, t - \tau) d\tau := \\ &:= \Psi(x, t) - \Phi(x, t, u), \end{aligned}$$

in which [5]

$$\begin{aligned} G(x, \xi, t) &= \frac{2a}{\gamma(l)(\xi^2 + a^2)^2} \sum_{k=1}^{\infty} \varphi_k(x) \varphi_k(\xi) \exp[-\alpha^2 \lambda_k^2 t], \quad \gamma(x) = \arctan \frac{x}{a}, \\ G_0(x, t) &= \int_0^l G(x, \xi, t) (\xi^2 + a^2)^2 \delta'(\xi) d\xi = -2\pi \sum_{k=1}^{\infty} k \varphi_k(x) \exp[-\alpha^2 \lambda_k^2 t], \\ \varphi_k(x) &= \sqrt{x^2 + a^2} \sin\left(\frac{\pi k}{\gamma(l)} \gamma(x)\right), \quad \lambda_k^2 = a^2 \left[\left(\frac{\pi k}{\gamma(l)}\right)^2 - 1 \right]. \end{aligned}$$

Thus, in this case we have to ensure the equality

$$\Phi(x, T, u) = \Psi(x, T), \quad x \in [0, l]. \quad (2.21)$$

Quadratic solution, satisfying consistency conditions is

$$u(t) = \frac{T-t}{T}\Theta_0(0) + u_1 t(T-t), \quad t \in [0, T],$$

and $u_1 = \text{const}$ must be determined in order to minimize (2.6) or make it small enough. In numerical implementation below we compare results obtained using this form and those— using (2.10).

Since the rod (domain of the problem) is finite, then Green's function and Fourier method give exactly the same expressions. Indeed, representing the solution of (2.18) as $\Theta(x, t) = \sum_{k=1}^{\infty} f_k(x)g_k(t)$, we obtain

$$\dot{g}_k(t) + \alpha^2 \lambda_k^2 g_k(t) = 0, \quad f_k''(x) + \frac{\lambda_k^2}{(x^2 + a^2)^2} f_k(x) = 0,$$

which implies

$$g_k(t) = c_{0k} \exp[-\alpha^2 \lambda_k^2 t]$$

and

$$f_k(x) = \sqrt{x^2 + a^2} \left[c_{1k} \sin \left(\sqrt{1 + \frac{\lambda_k^2}{a^2}} \gamma(x) \right) + c_{2k} \cos \left(\sqrt{1 + \frac{\lambda_k^2}{a^2}} \gamma(x) \right) \right].$$

Here c_{0k} , c_{1k} and c_{2k} are to be determined from boundary and initial conditions (2.19), (2.20):

$$f_k(0) = u_k, \quad f_k(l) = 0, \quad g_k(0) = \Theta_{0k},$$

where u_k are the coefficients of expansion of $u(t)$ by functions $g_k(t)$, and Θ_{0k} are the coefficients of $\Theta_0(x)$ by functions $f_k(x)$.

Example 4. Parameter optimization for time-dependent circuits

The last example that we would like to consider appears in study on non-linear time-dependent electrical oscillations in electronic circuits and allows exact satisfaction of terminal conditions. The phenomenon is described by Hill's differential equation in general (including Matthieu's equation as a particular case). In [20] it is solved for several circuit parameters in terms of Green's function. In particular, when circuit has parameters varying in time according to sawtooth law $u(t) = u_0 t + u_1$ with $u_0 \neq 0$, we have

$$\ddot{y} + u(t)y = 0, \quad 0 \leq t \leq T. \quad (2.22)$$

Suppose the input state is given $y(0) = y_0$, $y'(0) = y_{01}$ and one is interested in obtaining a specific output state $y(T) = y_T$, $y'(T) = y_{0T}$ with the right choice of parameters of the circuit.

Based on Green's function expression of (2.22) (see [20], eq.(19)), its general solu-

tion is given by

$$y(t) = y_0 \int_0^1 G(t, \tau) \delta'(\tau) d\tau + y_{01} \int_0^1 G(t, \tau) \delta(\tau) d\xi,$$

with

$$G(t, \tau) = -\frac{\pi}{3} u^{\frac{1}{2}}(t) u^{\frac{1}{2}}(\tau) \left[Y_{\frac{1}{3}}(\sigma(\tau)) J_{\frac{1}{3}}(\sigma(t)) - J_{\frac{1}{3}}(\sigma(\tau)) Y_{\frac{1}{3}}(\sigma(t)) \right],$$

$$\sigma(\tau) = \frac{2u^{\frac{3}{2}}(\tau)}{3u_0^2}.$$

This can be derived also from fundamental solutions of (2.22)– Airy functions [6].

Thus, in order to implement the condition $y(T) = y_T$, $y'(T) = y_{0T}$ one has to ensure the equalities

$$-y_0 \frac{\partial G(t, \tau)}{\partial \tau} \Big|_{\tau=0, t=T} + y_{01} G(T, 0) = y_T,$$

$$-y_0 \frac{\partial^2 G(t, \tau)}{\partial \tau \partial t} \Big|_{\tau=0, t=T} + y_{01} \frac{\partial G(t, \tau)}{\partial \tau} \Big|_{\tau=0, t=T} = y_{0T}.$$

Rescaling the time by factor T and assuming that all quantities are dimensionless (y is rescaled by y_0 , \dot{y} by y_{01} , u_0 by T^{-3} and u_1 by T^{-2}), numerical analysis is done and it is observed that there is a circuit parameter pair (u_0, u_1) for which the implemented $y(1)$, $y'(1)$ can be exactly equal to required (rescaled) y_T , y_{0T} (see Figure 1). The value of that pair is determined as the coordinates of the point of intersection of the contours $y(1) - \frac{y_T}{y_0} = 0$ and $y'(1) - \frac{y_{0T}}{y_{01}} = 0$ in the $u_0 u_1$ plane. Particularly, if $\frac{y_T}{y_0} = 0.5$, $\frac{y_{0T}}{y_{01}} = 1$, we obtain $u_0 = -17.56$ and $u_1 = 7.372$. In this case $u(t)$ changes its sign in $[0, 1]$.

Figure 2 shows the same contours for the case $\frac{y_T}{y_0} = 2$, $\frac{y_{0T}}{y_{01}} = 1$. Then $u_0 = -93.8$ and $u_1 = 123.8$, therefore $u(t) > 0$ for all $t \in [0, 1]$.

2.1 Numerical implementation

In order to justify the described procedure, we did numerical computations in Examples 2 and 3 considered above. In Example 2 we have introduced the dimensionless variables and functions

$$\tilde{x} = \frac{x}{x_0}, \quad \tilde{t} = \frac{t}{T}, \quad \tilde{\Theta} = \frac{\Theta}{\Theta^0}, \quad \tilde{u} = \frac{T}{\Theta^0 x_0} u, \quad \tilde{\alpha}^2 = \frac{T}{\Theta^0 x_0^2} \alpha^2,$$

in which $\Theta^0 = const$ is the reference temperature (for simplicity we take the intensity of initial temperature). Figures 3 and 4 shows the point plot of difference between required and implemented terminal states for quadratic and piecewise constant control laws ($\tilde{\alpha} = 1.515$). The maximal difference occurs at $\tilde{x} = 0.355$ and is equal to 0.0674

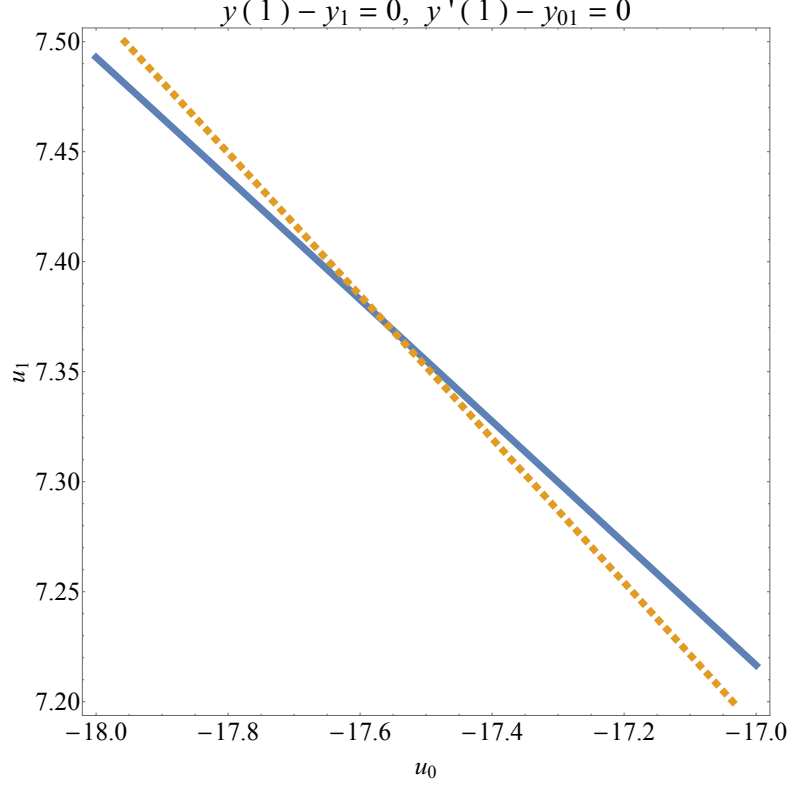


Figure 1: Contour plots of required and implemented states in the $u_0 u_1$ plane: $y(1) = \frac{y_T}{y_0}$ (solid line) and $y'(1) = \frac{y_{0T}}{y_{01}}$ (dashed line)

and at $\tilde{x} = 0.36$ and is equal to 0.06, correspondingly.

In Example 3 we have introduced the dimensionless variables and quantities

$$\tilde{x} = \frac{x}{l}, \quad \tilde{t} = \frac{t}{T}, \quad \tilde{\Theta} = \frac{\Theta}{\Theta^0}, \quad \tilde{u}_0 = \frac{u_0}{\Theta^0}.$$

The suggested procedure was implemented in Wolfram Mathematica 10.3 for entries $a = 1$, $\Theta_0(x) = \cos(\pi x)$, then $u(t) = (1 + u_1 t)(1 - t)$ (we omit $\tilde{\cdot}$ over dimensionless expressions). The series in Green's function expression is limited by 200 terms.

Numerical computations showed that $\Psi(x, T) \equiv 0$ in $x \in [0, 1]$, and for $u_1 = -0.5$

$$\max_{x \in [0, 1]} |\Phi(x, T, u) - \Psi(x, T)| = |\Phi(x, T, u) - \Psi(x, T)|_{x=0.5} = 10^{-4},$$

which decreases when x approaches the boundaries. Due to initial distribution of temperature It seems to be physically reasonable. One can manipulate further with u_1 to make the error smaller, but on this stage it is already satisfactory. For instance, when $u_1 = -1$, the error is of 10^{-6} order, and it decreases with decrease of u_1 (from

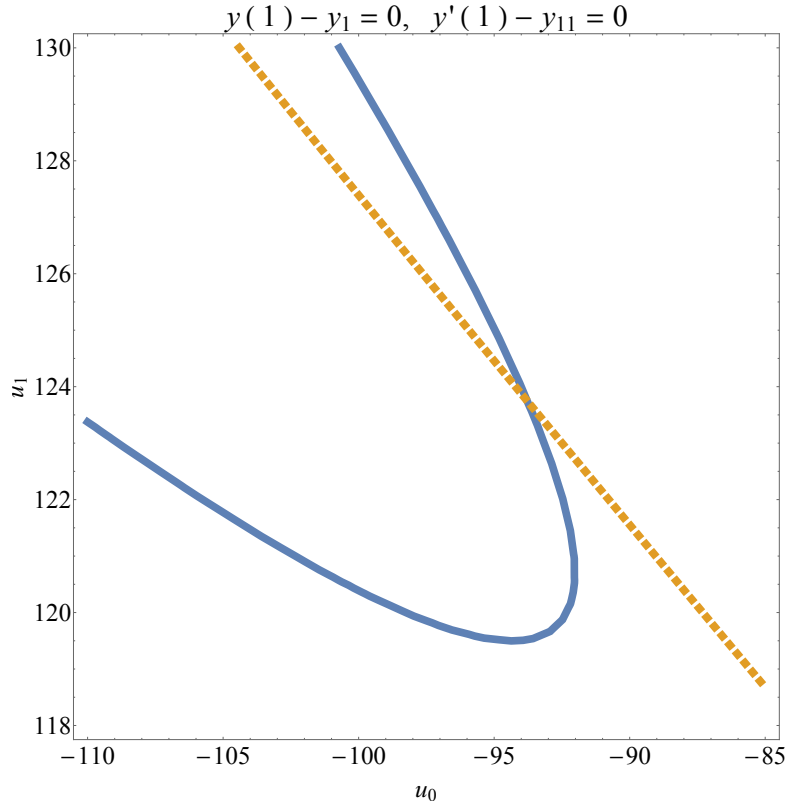


Figure 2: Contour plots of required and implemented states in the u_0u_1 plane: $y(1) = \frac{y_T}{y_0}$ (solid line) and $y'(1) = \frac{y_{0T}}{y_{01}}$ (dashed line)

-0.5 in negative direction) and decreases with increase of u_1 (from -0.5 in positive direction).

The same tendency is seen for $0 < a < 1$ and several other initial conditions like $\Theta_0(x) = \sin(\pi x)$ or $\tan(\pi x)$, etc., implying $u(t) = u_1 t(1 - t)$.

Suppose now $a = 0.5$, $\Theta_0(x) = 1 - x$, then $u(t) = (1 + u_1 t)(1 - t)$ (admission conditions are satisfied) and the implementation of (2.21) is plotted in Figure 5. The error is of 10^{-6} order, which in this stage is satisfactory.

Figure 5 shows how efficient the boundary quadratic heating regime is for a particular system. Let us now find the characteristics of boundary heating regime (2.10) minimizing (more precisely– making small enough) the error of approximation. We seek the control in the form $u(t) = \beta \sin(\omega t + \epsilon)$. In Figure 6 (2.21) is plotted when $6\omega = 5\pi$, $6\epsilon = \pi$ and $\beta = -0.3165$ (admission conditions will be satisfied when $\Theta_0(x)$ has a multiplier $\frac{\beta}{2}$). The refinement is done with manipulating the frequency ω (mainly), even though it can be done either by β and ϵ or both. During computations we keep 200, 250 and 500 terms in the expression for Green's sum, and the result is

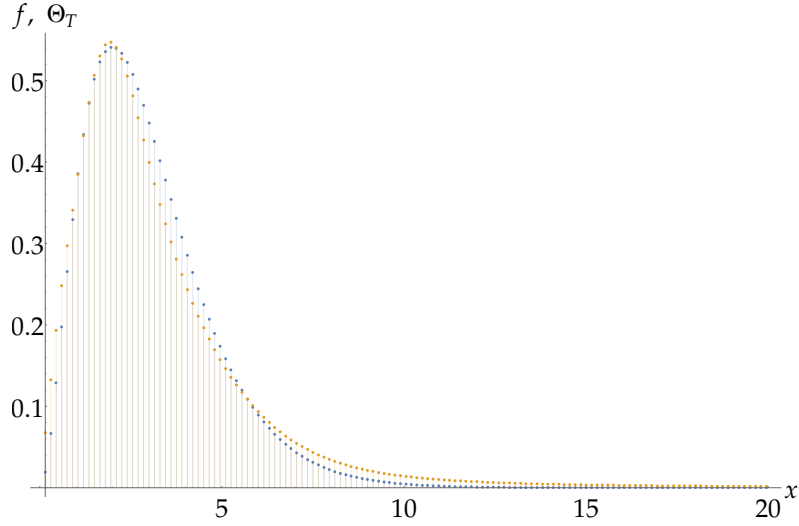


Figure 3: The left (with greater initial value) and right sides of (2.17) for $\tilde{\Theta}_0(\tilde{x}) = \frac{\tilde{x}}{1 + \tilde{x}^2}$, $\tilde{\Theta}_T(\tilde{x}) = \tilde{x}^2 \exp[-0.5\tilde{x}]$ and quadratic control

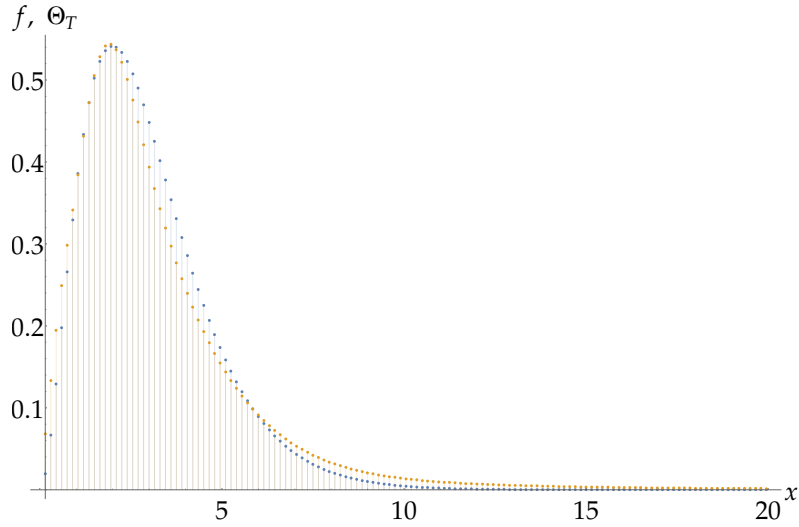


Figure 4: The left (with greater initial value) and right sides of (2.17) for $\tilde{\Theta}_0(x) = \tilde{x}(1 + \tilde{x}^2)^{-1}$, $\tilde{\Theta}_T(\tilde{x}) = \tilde{x}^2 \exp[-0.5\tilde{x}]$ and piecewise constant control

almost the same.

Even though the error in the case of quadratic boundary heating is less than that in the last case, it takes much less computational time in the second case than in the first one: in this sense the second boundary heating regime is efficient.

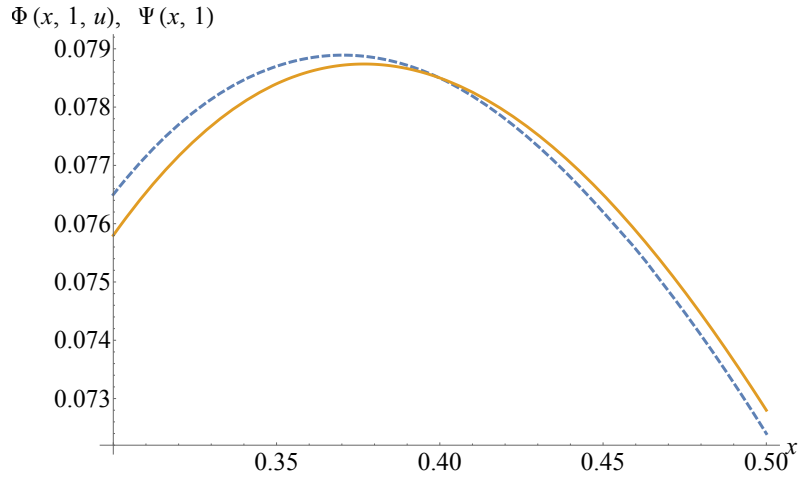


Figure 5: The graphs of $\Phi(x, 1, u)$ (dashed) and $\Psi(x, 1)$ (thick) for $u(t) = (1 - 1.3412t)(1 - t)$

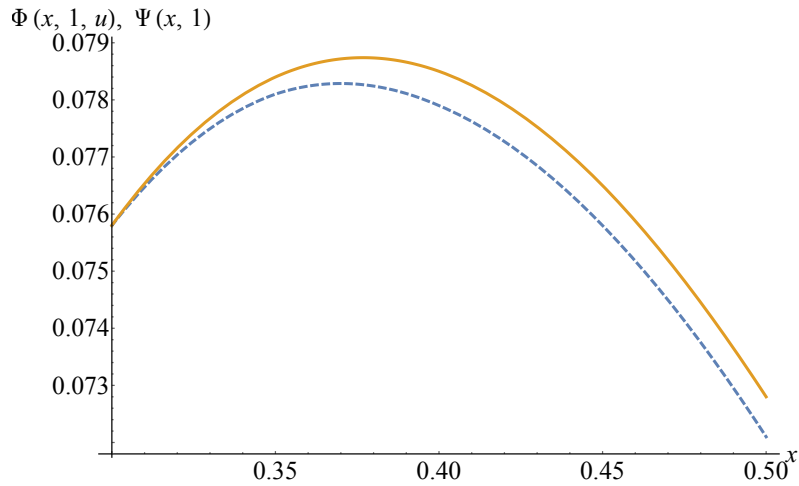


Figure 6: The graphs of $\Phi(x, 1, u)$ (dashed) and $\Psi(x, 1)$ (thick) for $u(t) = -0.3165 \sin\left(\frac{5\pi}{6}t + \frac{\pi}{6}\right)$

Conclusion

Thus, Green's function approach provides numerically efficient mathematical tool for ensuring approximate controllability for particular systems and, at the same time, algorithm for corresponding controls derivation. The procedure is explained as for processes taking place in unbounded domains, as well as those taking place in finite ones. It is a useful tool to handle as distributed, as well as boundary control problems.

It provides explicit formulas which simplify qualitative and quantitative analysis of control system under investigation. Numerical implementation of the algorithm in semi-infinite and finite domains showed its efficiency. For a particular form solution the approximation error turned to be of 10^{-6} order when we keep 200 terms in Green's series (Example 3).

Consistency of the approach with d'Alembert's and Fourier's methods is shown on particular examples.

The algorithm can be applied for much more complicated types of equations, their coupled systems arising in various applied problems: continuum mechanics, thermo- and electro-elasticity, wave-guides theory, to investigate control and optimization problems for non-homogeneous wave-guides and wave-guides with rough boundaries, etc. Further simplification of its implementability in general is needed.

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