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ИЗВЕСТИЯ НАЦИОНАЛЬНОЙ АКАДЕМИИ НАУК АРМЕНИИ

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THE PROBLEM OF THE OPTIMAL STABILIZATION OF THE SPINNING TOP MOTION

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Բանալի բառեր՝ դինամիկ համակարգեր, օպտիմալ ղեկավարում, օպտիմալ ստաբիլացում

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Задача оптимальной стабилизации вращательного движения волчка

В настоящей работе рассматривается задача оптимальной стабилизации вращательного движения волчка в линейном приближении. По направлениям соответствующих обобщенных координат введенны управляющие воздействия, проверена полная управляемость линейного приближения полученной системы управления. Решена задача оптимальной стабилизации этой системы на классическом смысле. Потом предполагается, что на волчок на конечном интервале времени действуют интегрально малые возмущающие силы. Поставлена и решена задача оптимальной стабилизации и в этом случае. Для обоих случаев построены оптимальные функции Ляпунова и оптимальные управляющие воздействия, получены оптимальные значения минимизируемых функционалов. Сравнение оптимальных значений минимизируемых функционалов показало, что затраченная энергия в случае оптимальной стабилизации при интегрально малых возмущениях меньше, чем при решении этой задачи в классическим методом.

Շահինյան Ս.Գ., Ռեզայի Մ.

Հոլի պտտական շարժման օպտիմալ ստաբիլացումը

Աշխատանքում դիտարկվում է հոլի պտտական շարժման օպտիմալ ստաբիլացման խնդիրը գծային մոտավորությամբ: Համապատասխան ընդհանրացված կոորդինատների ուղղությամբ ներմուծելով ղեկավարող ազդեցություններ, ստուգված է ստացված ղեկավարվող համակարգի գծային մոտավորության լրիվ ղեկավարելիությունը և լուծված է այդ համակարգի համար օպտիմալ ստաբիլացման խնդիրը: Այնուհետև ենթադրված է, որ հոլի վրա ժամանակի վերջավոր միջակայքում ազդում են ինտեգրալով փոքր գրգռող ուժեր: Ձևակերպված և լուծված է նաև օպտիմալ ստաբիլիզացիայի խնդիրը այդ դեպքում: Երկու դեպքերի համար էլ կառուցված են Լյապունովի օպտիմալ ֆունկցիաները և օպտիմալ ղեկավարող ազդեցությունները՝կախված համակարգի պարամետրերից: Ստացված են մինիմիզացվող ֆունկցիոնալների օպտիմալ արժեքները, որոնց համեմատությունը ցույց է տվել, որ ինտեգրալով փոքր գրգռումների դեպքում օպտիմալ ստաբիլացման ժամանակ ծախսվող էներգիան ավելի փոքր է, քան դասական իմաստով այդ խնդիրը լուծելիս:

The present work considers the optimal stabilization problem in motion of a Spinning Top when integrally small perturbations act during a finite interval of time. The optimal stabilization problem of considered motion is assumed and solved. In direction of the generalized coordinates introduced input controls, fully controllability of linear approximation of the obtained control system is checked up and the optimal stabilization problem of this system on classical sense is solved. Then, the problem will be limited to one input control, it is shown that the considered system is not fully controllable and for this case the optimal stabilization problem under integrally small perturbations of mentioned system is solved. For both cases optimal Lyapunov function is constructed, the optimal controls and the optimal value of performance index are obtained. The comparison between the optimal values of performance indexes proves that energy consumption at stabilization under integrally small perturbations is less than solving that issue in classical sense.

1.Introduction

Studies in the theory of optimal stabilization problem have begun from analytical design of regulators assumed by A.M. Letov [1-5]. Solution of the problem in this formulation is obtained by the classical variational method. To solve the problems of optimal stabilization, N.N. Krasovskii [6] proved the fundamental theorem of Lyapunov's second method, which is a connection method between Bellman dynamic programming [7] and the theorem of Lyapunov asymptotic stability. In [8], the problem of stabilizing controllers design in unstable motion of control system is considered. In [9, 10], the problem of stabilization of nonlinear control systems is studied. Like, the theory of stability, analogous to the theory of Lyapunov stability in the first approximation is developed [9], and, the problem of the minimization of the integral quality estimation for small initial perturbations is solved [10]. Rumyantsev [11] has carried out the task of optimal stabilization with respect to the variables, and has proved a theorem, that generalizes the fundamental theorem of optimal stabilization of all variables. In [12], the solution of stabilization problems and optimal stabilization of unstable motion of control system with sign-constant Lyapunov function usable state space has proved. Problem of stabilizing systems with alternate control is discussed in [13]. Using of alternate control imposes an additional condition for stabilizing control. It was shown, it should provide not only the asymptotic stability of the motion $q = 0$ but also the absence of sliding modes in the system $\dot{q} = F(q,u)$. Methods of studying stability and transition processes in linear stationary systems are investigated in [14].

It is well known that the circular movement of the top round its vertical axis will be stable if it rotates at a higher speed than a certain angular speed. The question is: how is it possible to provide the needed angular speed? In the present paper the optimal stabilization problem in motion of a spinning top as a rigid body is solved.

2. Problem Statement

Consider heavy spinning top, rotating around its dynamical symmetric axis with rotational (angular) velocity $\dot{\varphi}$. Let's two external forces applied on the spinning top only: the gravitational force $\vec{P}$ applied to the center of mass C of the spinning top, and the reaction $\vec{R}_0$ reliance on O (Fig. 1). Position of the dynamical symmetric axis z relative to the spinning top fixed axis $\xi\eta\zeta$ (vertical axis $O\zeta$) will define with the angles α and β [15] (Fig. 2).

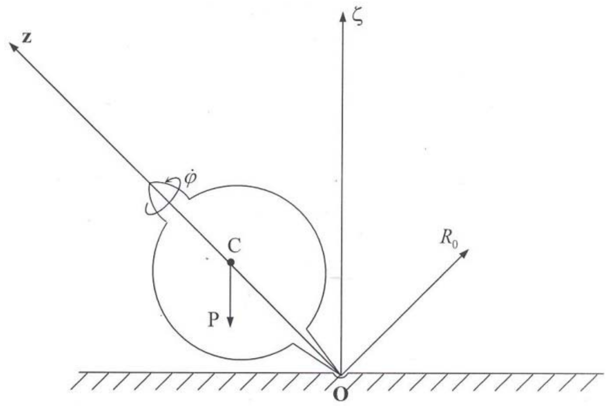

Fig. 1

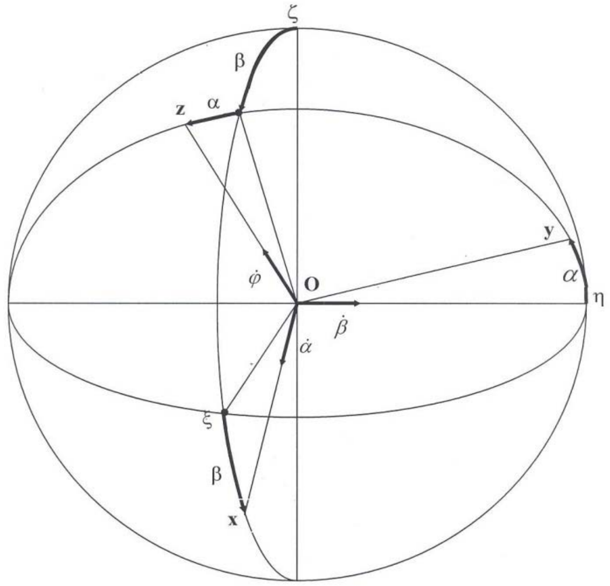

Fig. 2

We introduce the axes of x, y, z (Fig. 2) and p, q, r – the angular velocity about x, y, z axes, respectively, are defined by

$$\mathrm{p} = \dot{\alpha}, \quad \mathrm{q} = \dot{\beta}\cos\alpha, \quad \mathrm{r} = \dot{\varphi} - \dot{\beta}\sin\alpha. \tag{1}$$

Kinetic energy T and potential energy Π of system will be

$$T = \frac{1}{2} I_x \left(\dot{\alpha}^2 + \dot{\beta}^2 \cos^2 \alpha \right) + \frac{1}{2} I_z \left(\dot{\varphi} - \dot{\beta} \sin\alpha \right)^2,$$
$$\Pi = Pl \cos\alpha \cos\beta, \tag{2}$$

where l is distance between the centre of gravity of spinning top, C and the point O, I_x and I_z are moments of inertia of the spinning top around the axes x and z, respectively (as the spinning top is rotating and Oz is dynamical symmetric axis, $I_x = I_y$).

Now, we investigate the stability of the motion

$$\begin{gathered} \alpha = 0, \quad \dot{\alpha} = 0, \quad \beta = 0, \quad \dot{\beta} = 0, \\ \dot{\varphi} = \dot{\varphi}_0 = \omega = \mathrm{const}. \end{gathered} \tag{3}$$

Using of Lagrange equations [16], the system of differential equations of the spinning top motion will be

$$\begin{cases} \dfrac{d}{dt}\left(I_x \dot{\alpha}\right) + \dfrac{1}{2} I_x \dot{\beta} \sin 2\alpha + I_z \dot{\varphi} \dot{\beta} \cos\alpha - \\ -\dfrac{1}{2} I_z \dot{\beta}^2 \sin 2\alpha = Pl \sin\alpha \cos\beta, \\ \dfrac{d}{dt}\left(I_x \dot{\beta} \cos^2\alpha - I_z \left(\dot{\varphi} - \dot{\beta}\sin\alpha \right) \sin\alpha \right) = \\ = Pl \sin\beta \cos\alpha, \\ \dfrac{d}{dt}\left(I_z \left(\dot{\varphi} - \dot{\beta}\sin\alpha \right) \right) = 0. \end{cases} \tag{4}$$

Let's make following notations:

$$x_1 = \alpha, \quad x_2 = \dot{\alpha}, \quad x_3 = \beta,$$
$$x_4 = \dot{\beta}, \quad x_5 = \dot{\varphi} - \dot{\varphi}_0. \tag{5}$$

so, we obtain the first approximation of differential equations of the spinning top motion (4) in the form of $\dot{x} = A \cdot x$ as follows:

$$\begin{cases} \dot{x}_1 = x_2, \quad \dot{x}_2 = b \cdot x_1 - \sqrt{a} \cdot x_4, \\ \dot{x}_3 = x_4, \quad \dot{x}_4 = \sqrt{a} \cdot x_2 + b \cdot x_3, \quad \dot{x}_5 = 0, \end{cases} \tag{6}$$

where

$$a = \left(\frac{I_z}{I_x} \omega \right)^2, \quad b = \frac{Pl}{I_x}. \tag{7}$$

The characteristic equation for the system of differential equations (6) will be obtained as follows:

$$-\lambda\left(\lambda^4 + (a - 2b)\lambda^2 + b^2\right) = 0$$

so, we obtain

$$\left[\lambda_1 = 0, \quad \lambda^4 + (a - 2b)\lambda^2 + b^2 = 0. \right. \tag{8}$$

since $\mathrm{rank}A = 4$, and under the following condition:

$$\omega > \frac{2\sqrt{I_x Pl}}{I_z} \tag{9}$$

all the roots of the second equation of (8) are purely imaginary, so, the system of differential equations (6) is marginally stable in the sense of Lyapunov [17].

Let's ω is given rotational (angular) velocity. Let's consider the input controls $\overline{u}_1$ and $\overline{u}_2$ in the x_2 and x_5 generalized coordinate directions, respectively, so the spinning top motion (3) would become asymptotically stable. Then, the system of differential equations of the spinning top motion (6) will be

$$\begin{cases} \dot{x}_1 = x_2, \quad \dot{x}_2 = b \cdot x_1 - \sqrt{a} \cdot x_4 + \overline{u}_1, \\ \dot{x}_3 = x_4, \quad \dot{x}_4 = \sqrt{a} \cdot x_2 + b \cdot x_3, \quad \dot{x}_5 = \overline{u}_2. \end{cases} \tag{10}$$

Let's make following notations:

$$\begin{cases} y_1 = kx_1, \quad y_2 = \dfrac{1}{\sqrt{a}} x_2, \quad y_3 = kx_3, \\ y_4 = \dfrac{1}{\sqrt{a}} x_4, \quad y_5 = \dfrac{1}{\sqrt{a}} x_5; \\ u_1 = \dfrac{\overline{u}_1}{a}, \quad u_2 = \dfrac{\overline{u}_2}{\sqrt{a}}, \quad k = \dfrac{b}{a}, \quad t' = \sqrt{a}t. \end{cases} \tag{11}$$

Let's write the system of differential equations (10) in dimensionless form

$$\begin{cases} \dot{y}_1 = ky_2, \quad \dot{y}_2 = y_1 - y_4 + u_1, \\ \dot{y}_3 = ky_4, \quad \dot{y}_4 = y_2 + y_3, \quad \dot{y}_5 = u_2. \end{cases} \tag{12}$$

Here $\dot{y}_i = \frac{dy_i}{dt'}$.

That is, it is required that optimal controls u_1^0 and u_2^0 be found so that the system (12) would be asymptotically stable and the functional would acquire a minimal value.

It is quite easy to see that in case any of the input controls u_1^0 and u_2^0 is missing the problem will not be solved (the system will became a not fully controllability) and there is no need to introduce more directories. Actually, full controllability [6] of system of differential equations (12) can be checked easily, and turned out that it is full controllable as following calculations;

$$\mathrm{rank}K = \mathrm{rank}\left[B,\ AB,\ A^2B,\ A^3B,\ A^4B\right] = 5,$$

where

$$A = \begin{bmatrix} 0 & k & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & k & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Let's solve the problem of optimal stabilization of the system of differential equations (12) in the given sense in [6] while minimizing the performance index

$$I[u] = \int_0^\infty \left(\sum_{i=1}^{5} y_i^2 + \sum_{k=1}^{2} u_k^2 \right) dt. \tag{13}$$

3. Solution of the problem

Let's make up the expression [6]

$$B[u] = \sum_{i=1}^{5} \frac{\partial V}{\partial y_i} \dot{y}_i + \sum_{i=1}^{5} y_i^2 + \sum_{k=1}^{2} u_k^2 \tag{14}$$

since the expression (14) at optimal control takes the minimum value equal to zero [6], then

$$B\big|_{u=u^0} = 0, \tag{15}$$

and

$$\left.\frac{\partial B}{\partial u}\right|_{u_i=u_i^0} = 0, \quad (i = 1, 2) \tag{16}$$

where $u = (u_1 \quad u_2)^T$ and u_i^0 are optimal controls.

For Lyapunov function we will search in the form of

$$V = \frac{1}{2} \sum_{i,j=1}^{4} c_{ij} y_i y_j + \frac{1}{2} c_{55} y_5^2. \tag{17}$$

and c_{ij} − constants.

From equation (16) we obtain

$$u_1^0 = -\frac{1}{2} \frac{\partial V}{\partial y_2}; \quad u_2^0 = -\frac{1}{2} \frac{\partial V}{\partial y_5}. \tag{18}$$

By substituting the values u_i^0 of the equation (18) into equation (14), considering of equation (15), and using equation (17), we obtain the system of equations to define constants c_{ij} $(i, j = 1, \ldots, 5)$

$$\begin{cases} -\frac{1}{4}c_{12}^2 + c_{12} + 1 = 0, \quad \frac{k}{2}c_{11} + \frac{1}{2}c_{14} + \frac{1}{2}c_{22} - \frac{1}{4}c_{12}c_{22} = 0, \\ \frac{1}{2}c_{14} + \frac{1}{2}c_{23} - \frac{1}{4}c_{12}c_{23} = 0, \quad -\frac{1}{2}c_{12} + \frac{k}{2}c_{13} + \frac{1}{2}c_{24} - \frac{1}{4}c_{12}c_{24} = 0, \\ -\frac{1}{4}c_{22}^2 + kc_{12} + c_{24} + 1 = 0, \quad \frac{k}{2}c_{13} + \frac{1}{2}c_{24} + \frac{1}{2}c_{34} - \frac{1}{4}c_{22}c_{23} = 0, \\ \frac{k}{2}c_{14} - \frac{1}{2}c_{22} + \frac{k}{2}c_{23} + \frac{1}{2}c_{44} - \frac{1}{4}c_{22}c_{24} = 0, \\ -\frac{1}{4}c_{23}^2 + c_{34} + 1 = 0, \quad -\frac{1}{2}c_{23} + \frac{k}{2}c_{33} + \frac{1}{2}c_{44} - \frac{1}{4}c_{23}c_{24} = 0, \\ -\frac{1}{4}c_{24}^2 - c_{24} + kc_{34} + 1 = 0, \quad -\frac{1}{4}c_{55}^2 + 1 = 0. \end{cases} \tag{19}$$

The obtained solutions for constants c_{12}, c_{55} are independent from the value of k, and are listed below:

$$c_{12} = \pm 0{,}8284; \quad c_{55} = \pm 2{,}0000. \tag{20}$$

In order to obtaining the solutions for constants $c_{11}, c_{13}, c_{14}, c_{22}, c_{23}, c_{24}, c_{33}, c_{34}, c_{44}$, let's solve the system of equations (19) for various value of k (for example $k = 0.5, 1.0, 1.5, 2.0, \ldots, 50.0$), choose those solutions for which the Lyapunov's function becomes positively definite, then plot the graphs of mentioned constants vs. k. For example the graph of constant c_{11} vs. k is plotted below:

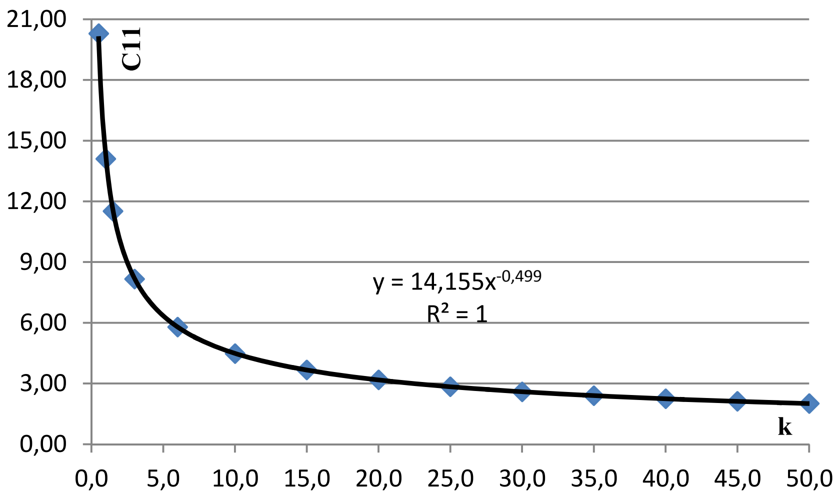

Fig. 3. Constant c_{11} vs. k and best estimated functions of constant c_{11} vs. k.

The dependence of constant c_{11} on k is displayed in fig. 3. The dots illustrate the solutions of c_{11} of the system (19) for corresponding k values, while the curve shows the best estimated functions of constant c_{11} vs. k.

Finally, for each values of k, we can similarly find the best estimated functions of constants $c_{11}^0, c_{13}^0, c_{14}^0, c_{22}^0, c_{23}^0, c_{24}^0, c_{33}^0, c_{34}^0, c_{44}^0$ vs. k, and we'll have;

$$
\begin{aligned}
&c_{11}^0 = 14.1550k^{-0.499}; \quad c_{22}^0 = 8.1943k^{0.5067}; \; c_{12}^0 = -0,8284;\\
&c_{13}^0 = \left(2\times10^{-8}\right)k^6 - \left(3\times10^{-6}\right)k^5 + 0.0002k^4 - 0.0074k^3 +\\
&+0.1226k^2 - 0.9883k - 22.8510;\\
&c_{14}^0 = \left(1\times10^{-7}\right)k^6 - \left(2\times10^{-5}\right)k^5 + 0.0012k^4 - 0.0408k^3 +\\
&+0.8058k^2 - 11.7640k - 14.7780;\\
&c_{23}^0 = 18.2040k^{0.5036}; \; c_{24}^0 = 18.4430k - 2.5861; c_{33}^0 = 84.0200k^{0.5029};\\
&c_{34}^0 = 85.0500k - 6.0128; \; c_{44}^0 = 82.5020k^{1.5082}; \; c_{55}^0 = 2,0000.
\end{aligned} \tag{21}
$$

Thus, optimal Lyapunov function will be

$$
\begin{aligned}
&V^0\left(y_1,\ldots,y_5\right) = \frac{c_{11}^0}{2}y_1^2 + \frac{c_{22}^0}{2}y_2^2 + \frac{c_{33}^0}{2}y_3^2 + \frac{c_{44}^0}{2}y_4^2 + y_5^2 -\\
&-0.8284y_1y_2 + c_{13}^0y_1y_3 + c_{14}^0y_1y_4 + c_{23}^0y_2y_3 + c_{24}^0y_2y_4 + c_{34}^0y_3y_4,
\end{aligned} \tag{22}
$$

and optimal controls will be

$$
\begin{aligned}
&u_1^0 = 0.4142y_1 - \frac{c_{22}^0}{2}y_2 - \frac{c_{23}^0}{2}y_3 - \frac{c_{24}^0}{2}y_4,\\
&u_2^0 = -y_5.
\end{aligned} \tag{23}
$$

For the optimal value of the performance index in equation (13) we obtain

$$
\begin{aligned}
&I^0 = V^0\left(y_{10},\ldots,y_{50}\right) = \frac{c_{11}^0}{2}y_{10}^2 + \frac{c_{22}^0}{2}y_{20}^2 + \frac{c_{33}^0}{2}y_{30}^2 + \frac{c_{44}^0}{2}y_{40}^2 + y_{50}^2 -\\
&-0.8284y_{10}y_{20} + c_{13}^0y_{10}y_{30} + c_{14}^0y_{10}y_{40} + c_{23}^0y_{20}y_{30} +\\
&+c_{24}^0y_{20}y_{40} + c_{34}^0y_{30}y_{40},
\end{aligned} \tag{24}
$$

where $y_{i0} = y_i(0) \quad (i = 1,\ldots,5)$.

4. Second Problem and its Solution

Consider again the system of differential equations (12). We assume that $u_1 = u; \quad u_2 = 0.$ Then the system of differential equations (12) will obtain

$$
\begin{cases}
\dot{y}_1 = ky_2, & \dot{y}_2 = y_1 - y_4 + u,\\
\dot{y}_3 = ky_4, & \dot{y}_4 = y_2 + y_3, \quad \dot{y}_5 = 0.
\end{cases} \tag{25}
$$

Let's replace the following problem: Finding such input control u^0 which will ensure the stability of the solution $y_i = 0 \quad (i = 1,\ldots,5)$ of the system of differential equations (25) under integrally small perturbations [18] and will minimize the performance index.

As the following calculations the system of differential equations (25) is not fully controllable;

$$\operatorname{ran}kK = \operatorname{ran}k\left[B_1, AB_1, A^2B_1, A^3B_1, A^4B_1\right] = 4,$$

Where $B_1 = [0 \quad 1 \quad 0 \quad 0 \quad 0]^T$

Hence, the optimal stabilization problem for the system of differential equations (25) in the sense [6] is not solved.

The system of differential equations (25) may solve the optimal stabilization problem under integrally small perturbations [18]. Minimized performance index should be adopted in the form of

$$I_1[u] = \int_0^\infty \left(\sum_{i=1}^4 y_i^2 + u^2 \right) dt. \tag{26}$$

Thus, it is required to resolve the optimal stabilization problem under integrally small perturbations for the system of differential equations (25) while minimizing the performance index in equation (26).

The expression of Bellman for the system of differential Equations (25) in this case will be

$$\begin{aligned} B[u] = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial y_1} ky_2 + \frac{\partial V}{\partial y_2}(y_1 - y_4 + u) + \frac{\partial V}{\partial y_3} ky_4 + \\ + \frac{\partial V}{\partial y_4}(y_2 + y_3) + y_1^2 + y_2^2 + y_3^2 + y_4^2 + u^2, \end{aligned} \tag{27}$$

since the expression in Equation (27) at optimal control takes the minimum value equal to zero [18], then

$\left.\dfrac{\partial B}{\partial u}\right|_{u=u^0} = \dfrac{\partial V}{\partial y_2} + 2u^0 = 0,$ so we obtain

$$u^0 = -\frac{1}{2}\frac{\partial V}{\partial y_2}. \tag{28}$$

As

$$B|_{u=u^0} = 0, \tag{29}$$

by substituting the value u^0 of the equation (28) into equation (29), we obtain

$$\begin{aligned} \frac{\partial V}{\partial t} + \frac{\partial V}{\partial y_1} ky_2 + \frac{\partial V}{\partial y_2}(y_1 - y_4) + \frac{\partial V}{\partial y_3} ky_4 + \frac{\partial V}{\partial y_4}(y_2 + y_3) + \\ + y_1^2 + y_2^2 + y_3^2 + y_4^2 - \frac{1}{4}\left(\frac{\partial V}{\partial y_2}\right)^2 = 0, \end{aligned} \tag{30}$$

For Lyapunov function we will search in the following form [18]:

$$V(t, y) = V_2(y) + V_1(t, y) + V_0(t), \tag{31}$$

where

$V_2(y)-$ the quadratic form with constant coefficients;

$V_1(t, y)-$ the first degree form with respect to y with the coefficients depending on time t;

$V_0(t)-$ the function of time t.

By substituting the value $V(t,y)$ of the expression (31) into equation (30), we obtain $V_1 = V_0 = 0$, and the equation (30) can be written as

$$\frac{\partial V_2}{\partial y_1} ky_2 + \frac{\partial V_2}{\partial y_2}(y_1 - y_4) + \frac{\partial V_2}{\partial y_3} ky_4 + \frac{\partial V_2}{\partial y_4}(y_2 + y_3) + \\ + y_1^2 + y_2^2 + y_3^2 + y_4^2 - \frac{1}{4}\left(\frac{\partial V_2}{\partial y_2}\right)^2 = 0, \tag{32}$$

Function $V_2(y)$ we can search in the form of $V_2(y) = \frac{1}{2}\sum_{i,j=1}^{4} c_{ij} y_i y_j$. Then we can find the solutions for constants $c_{11}, c_{12}, c_{13}, c_{14}, c_{22}, c_{23}, c_{24}, c_{33}, c_{34}, c_{44}$, which are similar to (20) and (21) exclusive of c_{55}.

Thus, optimal Lyapunov function will be

$$V^0(y_1, \ldots, y_4) = \frac{c_{11}^0}{2} y_1^2 + \frac{c_{22}^0}{2} y_2^2 + \frac{c_{33}^0}{2} y_3^2 + \frac{c_{44}^0}{2} y_4^2 - 0.8284 y_1 y_2 + \\ + c_{13}^0 y_1 y_3 + c_{14}^0 y_1 y_4 + c_{23}^0 y_2 y_3 + c_{24}^0 y_2 y_4 + c_{34}^0 y_3 y_4, \tag{33}$$

and optimal controls will be

$$u^0 = 0.4142 y_1 - \frac{c_{22}^0}{2} y_2 - \frac{c_{23}^0}{2} y_3 - \frac{c_{24}^0}{2} y_4. \tag{34}$$

For the optimal value of the performance index in equation (26) we obtain

$$I_1^0 = V^0(y_{10}, \ldots, y_{40}) = \frac{c_{11}^0}{2} y_{10}^2 + \frac{c_{22}^0}{2} y_{20}^2 + \frac{c_{33}^0}{2} y_{30}^2 + \frac{c_{44}^0}{2} y_{40}^2 - \\ - 0.8284 y_{10} y_{20} + c_{13}^0 y_{10} y_{30} + c_{14}^0 y_{10} y_{40} + c_{23}^0 y_{20} y_{30} + c_{24}^0 y_{20} y_{40} + c_{34}^0 y_{30} y_{40}, \tag{35}$$

where $y_{i0} = y_i(0) \quad (i = 1, \ldots, 4)$.

5. Conclusion

In the present work solved the optimal stabilization problem in motion of a Spinning Top. For constructing the solution in direction of the generalized coordinates introduced input controls, fully controllability of linear approximation of the obtained control system is checked up and the optimal stabilization problem of this system on classical sense is solved. Then, considers the optimal stabilization problem in motion of a Spinning Top when integrally small perturbations act during a finite interval of time. The optimal stabilization problem of considered motion is assumed and solved too. For both cases optimal Lyapunov function is constructed, the optimal controls and the optimal value of performance index are obtained.

A comparison between the values in Equation (24) and Equation (35) of the performance indexes in Equation (13) and Equation (26) has shown that $I_1^0 < I^0$.

It shows that energy consumption in stabilization at the given sense in [6] is more than stabilization under integrally small perturbations [18].

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