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ИЗВЕСТИЯ НАЦИОНАЛЬНОЙ АКАДЕМИИ НАУК АРМЕНИИ

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NON –AXIS – SYMMETRICAL DYNAMIC PROBLEM OF ELECTRO ELASTICITYFOR THE AXIALLY – POLARIZED PIEZOCERAMIC CYLINDER

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Key words: a problem of electro elasticity, a piezoceramic cylinder, a non-axis-symmetrical dynamic load.
Ключевые слова: задача электроупругости, пьезокерамический цилиндр, неосесимметричная динамическая нагрузка.
Բանալի բառեր. Էլեկտրաառաձգականության խնդիր, պիեզոկերամիկ գլան, ոչ առանցքասիմետրիկ դինամիկ բեռ

Շլյախին Դ.Ա.
Անիզոտրոպ պիեզոկերամիկ առանցքային բևեռացված գլանի համար ոչ առանցքասիմետրիկ խնդիր

Դիտարկված է ուղիղ պյեզոէֆեկտի ոչ առանցքասիմետրիկ խնդիր անիզոտրոպ պիեզոկերամիկ առանցքային բևեռացված գլանի համար երբ ճակատային մակերևույթների վրա ազդում են կամայական նորմալ լարումներ՝ կախված շառավղային, անկյունային կոորդինատներից և ժամանակից. Կառուցվել է նոր փակ լուծում սեփական վեկտոր-ֆունկցիաների վերլուծության միջոցով: Ստացված արտահայտություններ թույլ են տալիս որոշելու ոչ առանցքասիմետրիկ տատանումների հաճախականությունները և ինդուկցված էլեկտրական դաշտի բոլոր պարամետրերը.

Шляхин Д.А.
Неосесимметричная динамическая задача прямого пьезоэффекта для анизотропного пьезокерамического аксиально поляризованного цилиндра

Рассматривается неосесимметричная динамическая задача прямого пьезоэффекта для анизотропного пьезокерамического аксиально поляризованного цилиндра при действии на торцевых поверхностях нормальных напряжений, являющихся произвольными функциями радиальной, угловой координат и времени. Новое замкнутое решение построено методом разложения по собственным вектор-функциям в форме структурного алгоритма конечных преобразований. Полученные соотношения позволяют определять частоты собственных неосесимметричных колебаний, напряжённо-деформированное состояние элемента, а также все параметры индуцируемого электрического поля.

We consider a non-axis-symmetrical dynamic problem of the direct piezoeffect for anisotropic piezoceramic axially-polarized cylinder when normal stresses being arbitrary functions of the radial and angular coordinates and time act on the end surfaces. A new closed solution is constructed by the method of decomposition in terms of vector-functions in the form of a structural algorithm of finite transformations. The obtained expressions allow us to determine the frequencies of natural non-axis-symmetrical vibrations, the stress-strain state of the element and the parameters of the induced electric field.

Introduction

The most common structural elements of piezoceramic transducers are canonical bodies in the form of solid cylinders of finite size (thick circular plates). To describe their operation in real conditions and the enhanced functionality the need for the deeper analyses of time-varying processes without which it is impossible to understand the effect of the interaction of mechanical and electrical stress fields arises. However, existing methods of calculation of the piezoelectric elements of structures with reference to non-stationary effects are far from being perfect and most of them are approximate, but much of the research is associated with the development of numerical [1,2] and approximate [3-5] methods of solution and also bringing these problems to static [6,7].

In this regard, methods making possible obtaining closed solutions of unsteady initial-boundary problems of elasticity theory for finite bodies in three-dimensional formulation are of prime consideration now. With their help you can perform qualitative and quantitative assessment of the coupling of electromechanical stress fields in the piezoceramic elements of designs.

In the paper presented the dynamic problem is investigated using a consistent application of the finite integral transformations on all spatial variables. This approach allows us to obtain accurate, in the framework of the used models, estimated ratios in the most general form for the test piezoceramic cylinder.

1.The problem formulation. The solid anisotropic cylinder occupies the area Ω: $\{0 \le r_* \le b,\ 0 \le \theta \le 2\pi,\ 0 \le z_* \le h\}$ in the cylindrical coordinate system (r_*, θ, z_*) and it is made of a piezoceramic material with of a hexagonal system class 6mm[1] in which the axis of symmetry is parallel to the axial coordinate.The end membrane-anchored surfaces with electrodes ($z_* = 0, h$) are under the arbitrary dynamic load (normal stresses) $q_1^*(r_*, \theta,\ t_*)$, $q_2^*(r_*, \theta,\ t_*)$ and connected to the measuring device with a high input resistance, what corresponds to the "idling mode". Various mechanical conditions can be satisfied on cylindrical surfaces without electrodes. For the sake of definiteness we will consider them free from normal and tangential stresses. In this formulation the problem simulates the operation of the piezoelectric elements in the devices of the direct piezoelectric effect, transforming a mechanical effect to the corresponding electric signal.

The mathematical formulation of the given problem of electro elasticity in the dimensionless form includes a system of differential equations in relation to the components of the displacement vector $U(r,\theta,z,t), V(r,\theta,z,t), W(r,\theta,z,t)$, the potential of the electric field $\varphi(r,\theta,z,t)$ and the initial-boundary conditions [8]:

$$\nabla_1^2 U + a_1 \frac{1}{r^2}\frac{\partial^2 U}{\partial\theta^2} + a_2 \frac{\partial^2 U}{\partial z^2} + a_3 \frac{1}{r}\frac{\partial^2 V}{\partial r \partial\theta} - a_4 \frac{1}{r^2}\frac{\partial V}{\partial\theta} + a_5 \frac{\partial^2 W}{\partial r \partial z} + a_6 \frac{\partial^2 \varphi}{\partial r \partial z} - \frac{\partial^2 U}{\partial t^2} = 0 \quad (1.1)$$

$$a_1 \nabla_1^2 V + \frac{1}{r^2}\frac{\partial^2 V}{\partial\theta^2} + a_2 \frac{\partial^2 V}{\partial z^2} + a_3 \frac{1}{r}\frac{\partial^2 U}{\partial r \partial\theta} + a_4 \frac{1}{r^2}\frac{\partial U}{\partial\theta} + a_5 \frac{1}{r}\frac{\partial^2 W}{\partial\theta \partial z} + a_6 \frac{1}{r}\frac{\partial^2 \varphi}{\partial\theta \partial z} - \frac{\partial^2 V}{\partial t^2} = 0$$

$$a_5 \frac{\partial}{\partial z}\left(\nabla U + \frac{1}{r}\frac{\partial V}{\partial\theta}\right) + a_2\left(\nabla_2^2 W + \frac{1}{r^2}\frac{\partial^2 W}{\partial\theta^2}\right) + a_7 \frac{\partial^2 W}{\partial z^2} + a_8\left(\nabla_2^2\varphi + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2}\right) + \frac{\partial^2\varphi}{\partial z^2} - \frac{\partial^2 W}{\partial t^2} = 0$$

$$a_6 \frac{\partial}{\partial z}\left(\nabla U + \frac{1}{r}\frac{\partial V}{\partial\theta}\right) + a_8\left(\nabla_2^2 W + \frac{1}{r^2}\frac{\partial^2 W}{\partial\theta^2}\right) + \frac{\partial^2 W}{\partial z^2} - a_9\left(\nabla_2^2\varphi + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2}\right) - a_{10}\frac{\partial^2\varphi}{\partial z^2} = 0$$

$$z = 0, L \quad \sigma_{zz|z=0} = a_{11}\left(\nabla U + \frac{1}{r}\frac{\partial V}{\partial\theta}\right) + a_7 \frac{\partial W}{\partial z} + \frac{\partial\varphi}{\partial z} = q_1(r,\theta,t) \quad (1.2)$$

$$\sigma_{zz|z=L} = a_{11}\left(\nabla U + \frac{1}{r}\frac{\partial V}{\partial\theta}\right) + a_7 \frac{\partial W}{\partial z} + \frac{\partial\varphi}{\partial z} = q_2(r,\theta,t)$$

$$U(r,\theta,0,t) = U(r,\theta,L,t) = 0, \quad V(r,\theta,0,t) = V(r,\theta,L,t) = 0$$

$$D_{z|z=0,L} = -a_{10}\frac{\partial\varphi}{\partial z} + a_{12}\left(\nabla U + \frac{1}{r}\frac{\partial V}{\partial r}\right) + \frac{\partial W}{\partial z} = 0$$

$$\theta = 0, 2\pi \quad U(r,0,z,t) = U(r,2\pi n,z,t), \quad \frac{\partial U}{\partial\theta}_{|\theta=0} = \frac{\partial U}{\partial\theta}_{|\theta=2\pi n} \quad (1.3)$$

$$V(r,0,z,t) = V(r,2\pi n,z,t), \quad \frac{\partial V}{\partial\theta}_{|\theta=0} = \frac{\partial V}{\partial\theta}_{|\theta=2\pi n},$$

$$W(r,0,z,t) = W(r,2\pi n,z,t), \quad \frac{\partial W}{\partial\theta}_{|\theta=0} = \frac{\partial W}{\partial\theta}_{|\theta=2\pi n}$$

[1] Piezoceramics this class is a core material for manufacturing the conversion elements of different devices. It is explained how the parameters of these high piezoelectric materials and the ability to change their properties in a wide range by varying the molar concentration.

$$\varphi(r,0,z,t)=\varphi(r,2\pi n,z,t),\ \frac{\partial\varphi}{\partial\theta}_{|\theta=0}=\frac{\partial\varphi}{\partial\theta}_{|\theta=2\pi n}$$

$$r=1,0\quad \sigma_{rr|r=1}=\frac{\partial U}{\partial r}+a_{13}\frac{1}{r}\left(U+\frac{\partial V}{\partial\theta}\right)+a_{11}\frac{\partial W}{\partial z}+a_{12}\frac{\partial\varphi}{\partial z}=0 \tag{1.4}$$

$$\sigma_{rz|r=1}=a_2\left(\frac{\partial W}{\partial r}+\frac{\partial U}{\partial z}\right)+a_8\frac{\partial\varphi}{\partial r}=0,\ \sigma_{r\theta|r=1}=a_1\left[\frac{1}{r}\left(\frac{\partial U}{\partial\theta}-V\right)+\frac{\partial V}{\partial r}\right]=0$$

$$D_{r|r=1}=-a_9\frac{\partial\varphi}{\partial r}+a_8\left(\frac{\partial W}{\partial r}+\frac{\partial U}{\partial z}\right)=0$$

$$U(0,\theta,z,t)<\infty,\ V(0,\theta,z,t)<\infty,\ W(0,\theta,z,t)<\infty,\ \varphi(0,\theta,z,t)<\infty;$$

$$t=0\quad U(r,\theta,z,0)=U_0(r,\theta,z),\ \dot U(r,\theta,z,0)=\dot U_0(r,\theta,z), \tag{1.5}$$

$$V(r,\theta,z,0)=V_0(r,\theta,z),\ \dot V(r,\theta,z,0)=\dot V_0(r,\theta,z),$$

$$W(r,\theta,z,0)=W_0(r,\theta,z),\ \dot W(r,\theta,z,0)=\dot W_0(r,\theta,z);$$

where $\{U,V,W\}=\{U^*,V^*,W^*\}/b$, $\varphi=\varphi^* e_{33}/(bC_{11})$, $\{r,z,L\}=\{r_*,z_*,h\}/b$,

$t=t_* b^{-1}\sqrt{C_{11}/\rho}$, $\nabla_1^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}$, $\nabla_2^2=\nabla_1^2+\frac{1}{r^2}$, $\nabla=\frac{\partial}{\partial r}+\frac{1}{r}$,

$a_1=\frac{C_{66}}{C_{11}}$, $a_2=\frac{C_{55}}{C_{11}}$, $a_3=\frac{(C_{12}+C_{66})}{C_{11}}$, $a_4=\frac{(C_{11}+C_{66})}{C_{11}}$, $a_5=\frac{(C_{13}+C_{55})}{C_{11}}$, $a_6=\frac{(e_{31}+e_{15})}{e_{33}}$,

$a_7=\frac{C_{33}}{C_{11}}$, $a_8=\frac{e_{15}}{e_{33}}$, $a_9=\frac{C_{11}\varepsilon_{11}}{e_{33}^2}$, $a_{10}=\frac{C_{11}\varepsilon_{33}}{e_{33}^2}$, $a_{11}=\frac{C_{13}}{C_{11}}$, $a_{12}=\frac{e_{31}}{e_{33}}$, $a_{13}=\frac{C_{12}}{C_{11}}$,

$\{q_1,q_2\}=\{q_1^*,q_2^*\}/C_{11}$; t_* – time; ρ, C_{ms}, e_{ms} – bulk density, electric constants and piezoelectric modules of the anisotropic piezoceramic material ($m,s=1,6$); $\varepsilon_{11},\varepsilon_{33}$ – dielectric permeability in the radial and axial directions; U^*,V^*,W^*,φ^* – the components of the vector of displacements and the potential of the electric field in a dimensional form; $U_0,\dot U_0$, $V_0,\dot V_0$, , $W_0,\dot W_0$ – displacements and their velocities known at the initial moment of time.

Equations (1.3) and (1.4) at $r=0$ are the conditions of periodicity for circular areas and regularity of solutions. In equations (1.5) and those below the dot means differentiation by t

2. The construction of the general solution. The solution is made by the method of integral transformations consistently using the sine and cosine Fourier transform with finite limits on variables θ and z , as well as a generalized finite transformation [8] on the radial coordinate r. Each time first you must perform the procedure of standardization (harmonization of boundary conditions on the corresponding homogeneous coordinate). At the first stage the following representation is used for this purpose:

$$\{U(r,\theta,z,t),V(r,\theta,z,t)\}=H_1(r,\theta,z,t)+\{u(r,\theta,z,t),v(r,\theta,z,t)\} \tag{2.1}$$

$$W(r,\theta,z,t)=H_2(r,\theta,z,t)+w(r,\theta,z,t)$$

$$\varphi(r,\theta,z,t)=H_3(r,\theta,z,t)+\chi(r,\theta,z,t)$$

Here $H_1=(Lz-z^2)(q_1+q_2)$, $H_3=a_9^{-1}H_2$

$$H_2 = a_9 \left(a_{10} - 1\right)^{-1} \left[\left(z - \frac{z^2}{2L} - \frac{L}{2} \right) q_1 + \frac{z^2}{2L} q_2 \right].$$

The substitution of (2.1) for (1.1) – (1.5) gives a new initial boundary problem regarding functions $u(r,\theta,z,t)$, $v(r,\theta,z,t)$, $w(r,\theta,z,t)$, $\chi(r,\theta,z,t)$ with homogeneous boundary conditions on coordinates z and θ. As this takes place, differential equations (1.1) and boundary conditions (1.4) do not agree with $F_1 \div F_4$ and $N_1 \div N_4$, and the initial conditions (1.5) should be replaced by u_0, v_0, w_0.

To the transformed boundary value problem (1.1) – (1.5) we consistently apply the sine and cosine Fourier transform with finite limits on variables z and θ using the following transformants:

$$\left\{u_s\left(r,\theta,n,t\right), v_s\left(r,\theta,n,t\right)\right\} = \int_0^L \left\{u\left(r,\theta,z,t\right), v\left(r,\theta,z,t\right)\right\} \sin\left(j_n z\right) dz, \quad (2.2)$$

$$\left\{w_c\left(r,\theta,n,t\right), \chi_c\left(r,\theta,n,t\right)\right\} = \int_0^L \left\{w\left(r,\theta,z,t\right), \chi\left(r,\theta,z,t\right)\right\} \cos\left(j_n z\right) dz,$$

$$\left\{U_c\left(r,m,n,t\right), W_c\left(r,m,n,t\right), \varphi_c\left(r,m,n,t\right)\right\} =$$
$$= \int_0^{2\pi} \left\{u_s\left(r,\theta,n,t\right), w_c\left(r,\theta,n,t\right), \chi_c\left(r,\theta,n,t\right)\right\} \cos\left(m\theta\right) d\theta,$$

$$V_s\left(r,m,n,t\right) = \int_0^{2\pi} v_s\left(r,\theta,n,t\right) \sin\left(m\theta\right) d\theta,$$

with the appropriate conversion formulas

$$\left\{u_s\left(r,\theta,n,t\right), w_s\left(r,\theta,n,t\right), \chi_s\left(r,\theta,n,t\right)\right\} = \quad (2.3)$$
$$= \sum_{m=0}^{\infty} P_m^{-1} \left\{U_c\left(r,m,n,t\right), W_c\left(r,m,n,t\right), \varphi_c\left(r,m,n,t\right)\right\} \cos\left(m\theta\right),$$

$$v_s\left(r,\theta,n,t\right) = \pi^{-1} \sum_{m=1}^{\infty} V_s\left(r,m,n,t\right) \sin\left(m\theta\right),$$

$$\left\{u\left(r,\theta,z,t\right), v\left(r,\theta,z,t\right)\right\} = \frac{2}{L} \sum_{n=1}^{\infty} \left\{u_s\left(r,\theta,n,t\right), v_s\left(r,\theta,n,t\right)\right\} \sin j_n z,$$

$$\left\{w\left(r,\theta,z,t\right), \chi\left(r,\theta,z,t\right)\right\} = \sum_{n=0}^{\infty} \Omega_n^{-1} \left\{w_c\left(r,\theta,n,t\right), \chi_c\left(r,\theta,n,t\right)\right\} \cos j_n z,$$

$$j_n = n\pi / L, \quad P_m = \begin{cases} 2\pi, & (m = 0) \\ \pi, & (m \neq 0) \end{cases}, \quad \Omega_n = \begin{cases} L, & (n = 0) \\ L/2, & (n \neq 0) \end{cases}.$$

As a result we get the following initial-boundary problem concerning the Fourier transformants $U_c\left(r,m,n,t\right), V_s\left(r,m,n,t\right), W_c\left(r,m,n,t\right), \varphi_c\left(r,m,n,t\right)$:

$$\nabla_1^2 U_c - a_1 \frac{m^2}{r^2} U_c - a_2 j_n^2 U_c + a_3 \frac{m}{r} \frac{\partial V_s}{\partial r} - a_4 \frac{m}{r^2} V_s - a_5 j_n \frac{\partial W_c}{\partial r} - a_6 j_n \frac{\partial \varphi_c}{\partial r} - \frac{\partial^2 U_c}{\partial t^2} = R_{1c} \quad (2.4)$$

$$a_1 \nabla_1^2 V_s - \frac{m^2}{r^2} V_s - a_2 j_n^2 V_s - a_3 \frac{m}{r} \frac{\partial U_c}{\partial r} - a_4 \frac{m}{r^2} U_c + a_5 j_n \frac{m}{r} W_c + a_6 j_n \frac{m}{r} \varphi_c - \frac{\partial^2 V_s}{\partial t^2} = R_{2s}$$

$$a_5 j_n \left(\nabla U_c + \frac{m}{r} V_s \right) + a_2 \left(\nabla_2^2 W_c - \frac{m^2}{r^2} W_c \right) - a_7 j_n^2 W_c + a_8 \left(\nabla_2^2 \varphi_c - \frac{m^2}{r^2} \varphi_c \right) - j_n^2 \varphi_c - \frac{\partial^2 W_c}{\partial t^2} = R_{3c}$$

$$a_6 j_n \left(\nabla U_c + \frac{m}{r} V_s \right) + a_8 \left(\nabla_2^2 W_c - \frac{m^2}{r^2} W_c \right) - j_n^2 W_c - a_9 \left(\nabla_2^2 \varphi_c - \frac{m^2}{r^2} \varphi_c \right) + a_{10} j_n^2 \varphi_c = R_{4c},$$

$$r=1,0 \quad \frac{\partial U_c}{\partial r}+a_{13}\frac{1}{r}\left(U_c+mV_s\right)-a_{11}j_nW_c-a_{12}j_n\varphi_c=Y_{1c|r=1}, \tag{2.5}$$

$$a_2\left(\frac{\partial W_c}{\partial r}+j_nU_c\right)+a_8\frac{\partial\varphi_c}{\partial r}=Y_{2c|r=1}, \quad a_1\left[\frac{\partial V_s}{\partial r}-\frac{1}{r}\left(mU_c+V_s\right)\right]=Y_{3s|r=1},$$

$$-a_9\frac{\partial\varphi_c}{\partial r}+a_8\left(\frac{\partial W_c}{\partial r}+j_nU_c\right)=Y_{4c|r=1},$$

$$U_c\left(0,m,n,t\right)<\infty, \; V_s\left(0,m,n,t\right)<\infty, \; W_c\left(0,m,n,t\right)<\infty, \; \varphi_c\left(0,m,n,t\right)<\infty;$$

$$t=0 \quad U_c\left(r,m,n,0\right)=U_{0c}\left(r,m,n\right), \; \dot{U}_c\left(r,m,n,0\right)=\dot{U}_{0c}\left(r,m,n\right), \tag{2.6}$$

$$V_s\left(r,m,n,0\right)=V_{0s}\left(r,m,n\right), \; \dot{V}_s\left(r,m,n,0\right)=\dot{V}_{0s}\left(r,m,n\right),$$

$$W_c\left(r,m,n,0\right)=W_{0c}\left(r,m,n\right), \; W_c\left(r,m,n,0\right)=\dot{W}_{0c}\left(r,m,n\right);$$

where $\left\{R_{1c},R_{3c},R_{4c},Y_{1c},Y_{2c},Y_{4c},U_{0c},\dot{U}_{0c},W_{0c},\dot{W}_{0c}\right\}=$

$$=\int_0^{2\pi}\left\{F_{1s},F_{3c},F_{4c},N_{1s},N_{2c},N_{4c},u_{0s},\dot{u}_{0s},w_{0c},\dot{w}_{0c}\right\}\cos\left(m\theta\right)d\theta,$$

$$\left\{R_{2s},Y_{3s},V_{0s},V_{0s}\right\}=\int_0^L\left\{F_{2s},N_{3s},v_{0s},\dot{v}_{0s}\right\}\sin\left(m\theta\right)d\theta,$$

$$\left\{F_{1s},F_{2s},N_{1s},N_{3s},u_{0s},\dot{u}_{0s},v_{0s},\dot{v}_{0s}\right\}=\int_0^L\left\{F_1,F_2,N_1,N_3,u_0,\dot{u}_0,v_0,\dot{v}_0\right\}\sin\left(j_nz\right)dz,$$

$$\left\{F_{3c},F_{4c},N_{2c},N_{4c},w_{0c},\dot{w}_{0c}\right\}=\int_0^L\left\{F_3,F_4,N_2,N_4,w_0,\dot{w}_0\right\}\cos\left(j_nz\right)dz.$$

Standardizing the problem once more (2.4) – (2.6) we represent the Fourier transformants U_c,V_s,W_c,φ_c as follows:

$$U_c\left(r,m,n,t\right)=H_4\left(r,m,n,t\right)+U_c^*\left(r,m,n,t\right), \tag{2.7}$$

$$V_s\left(r,m,n,t\right)=H_5\left(r,m,n,t\right)+V_s^*\left(r,m,n,t\right),$$

$$W_c\left(r,m,n,t\right)=H_6\left(r,m,n,t\right)+W_c^*\left(r,m,n,t\right),$$

$$\varphi_c\left(r,m,n,t\right)=H_7\left(r,m,n,t\right)+\varphi_c^*\left(r,m,n,t\right),$$

where $H_4=\left(r-1\right)Y_{1c|r=1}$, $H_5=a_1^{-1}\left(r-1\right)Y_{3s|r=1}$, $H_6=a_2^{-1}\left(r-1\right)\left(Y_{2c|r=1}-a_8Y_{5c|r=1}\right)$,

$$H_7=\left(r-1\right)Y_{5c|r=1}, \; Y_{5c|r=1}=\frac{a_8}{a_2a_9+a_8^2}\left(Y_{4c|r=1}-\frac{a_2}{a_8}Y_{2c|r=1}\right).$$

The substitution of (2.7) for (2.4) – (2.6) gives the initial- boundary problem regarding functions $U_c^*,V_s^*,W_c^*,\varphi_c^*$ with homogeneous boundary conditions on the coordinate r. And it is necessary to take $R_{1c}^*,R_{2s}^*,R_{3c}^*,R_{2c}^*$ and $U_{0c}^*,\dot{U}_{0c}^*,V_{0s}^*,\dot{V}_{0s}^*,W_{0c}^*,\dot{W}_{0c}^*$ instead of the right-hand parts of differential equations $R_{1c},R_{2s},R_{3c},R_{4c}$ (2.4) andinitial conditions $U_{0c},\dot{U}_{0c},V_{0s},\dot{V}_{0s},W_{0c},\dot{W}_{0c}$ (2.6).

The initial – boundary problem (2.4) – (2.6) regarding functions $U_c^*,V_s^*,W_c^*,\varphi_c^*$ is solved using a structural algorithm of the generalized method of finite integral transformations (FIT) [9]. Enter into the segment [0,1] a generate FIT with the unknown components $K_1\left(\lambda_{inm},r\right),K_2\left(\lambda_{inm},r\right),K_3\left(\lambda_{inm},r\right),K_4\left(\lambda_{inm},r\right)$ of the vector-function of the kernel of transformations:

$$G(\lambda_{inm}, m, n, t) = \int_0^1 \left(U_c^* K_1 + V_s^* K_2 + W_c^* K_3 \right) r dr \, , \tag{2.8}$$

$$\left\{ U_c^*, V_s^*, W_c^*, \varphi_c^* \right\} = \sum_{i=1}^{\infty} G\left\{ K_1, K_2, K_3, K_4 \right\} \left\| K_{inm} \right\|^{-2}, \tag{2.9}$$

$$\left\| K_{inm} \right\|^2 = \int_0^1 \left[K_1^2 + K_2^2 + K_3^2 \right] r dr \, ,$$

where λ_{inm} – are positive parameters that form a countable set $\left(i = \overline{1, \infty}\right)$.

The equality (2.8) is a transformant and (2.9) are inversion formulas of the FIT method.

Circular frequencies of non-axis-symmetrical oscillations of the cylinder ω_{inm} are associated with the following function λ_{inm}

$$\omega_{inm} = \frac{\lambda_{inm}}{b} \sqrt{\frac{C_{11}}{\rho}} \, . \tag{2.10}$$

Subjecting a system of equations and conditions (2.4) – (2.6) under $U_c^*, V_s^*, W_c^*, \varphi_c^*$ functions to transformations in accordance with the structural algorithm [9] we get a countable set of Cauchy problems for the transformant $G(\lambda_{inm}, m, n, t)$, the solution of which has the following form:

$$G(\lambda_{inm}, m, n, t) = G_0 \cos(\lambda_{inm} t) + \dot{G}_0 \sin(\lambda_{inm} t) / \lambda_{inm} - \lambda_{inm}^{-1} \int_0^t F(\lambda_{inm}, m, n, t) \times \\ \times \sin \lambda_{inm} (t - \tau) \, d\tau. \tag{2.11}$$

and a homogeneous boundary problem for components K_1, K_2, K_3, K_4:

$$\left(\nabla_1^2 - a_1 \frac{m^2}{r^2} - a_2 j_n^2 + \lambda_{inm}^2 \right) K_1 + \left(a_3 \frac{m}{r} \frac{d}{dr} - a_4 \frac{m}{r^2} \right) K_2 - a_5 j_n \frac{dK_3}{dr} - a_6 j_n \frac{dK_4}{dr} = 0 \tag{2.12}$$

$$\left(a_1 \nabla_1^2 - \frac{m^2}{r^2} - a_2 j_n^2 + \lambda_{inm}^2 \right) K_2 - \left(a_3 \frac{m}{r} \frac{d}{dr} + a_4 \frac{m}{r^2} \right) K_1 + a_5 j_n \frac{m}{r} K_3 + a_6 j_n \frac{m}{r} K_4 = 0$$

$$a_5 j_n \left(\nabla K_1 + \frac{m}{r} K_2 \right) + \left[a_2 \left(\nabla_2^2 - \frac{m^2}{r^2} \right) - a_7 j_n^2 + \lambda_{inm}^2 \right] K_3 + \left[a_8 \left(\nabla_2^2 - \frac{m^2}{r^2} \right) - j_n^2 \right] K_4 = 0$$

$$a_6 j_n \left(\nabla K_1 + \frac{m}{r} K_2 \right) + \left[a_8 \left(\nabla_2^2 - \frac{m^2}{r^2} \right) - j_n^2 \right] K_3 - \left[a_9 \left(\nabla_2^2 - \frac{m^2}{r^2} \right) - a_{10} j_n^2 \right] K_4 = 0$$

$$r = 1 \quad \frac{dK_1}{dr} + a_{13} \left(K_1 + mK_2 \right) - a_{11} j_n K_3 - a_{12} j_n K_4 = 0 \tag{2.13}$$

$$\frac{dK_3}{dr} + j_n K_1 = 0 \, , \quad \frac{dK_4}{dr} = 0 \, , \quad \frac{dK_2}{dr} - mK_1 - K_2 = 0$$

$$r = 0 \quad K_1 < \infty \, , \; K_2 < \infty \, , \; K_3 < \infty \, , \; K_4 < \infty \, .$$

Here $F(\lambda_{inm}, m, n, t) = \int_0^1 \left[R_{1s}^* K_1 + R_{2c}^* K_2 + R_{3s}^* K_3 + R_{4s}^* K_4 \right] \cdot r dr$

$$G_0(\lambda_{inm}, m, n) = \int_0^1 \left[U_{0c}^* K_1 + V_{0s}^* K_2 + W_{0c}^* K_3 \right] \cdot r dr$$

$$\dot{G}_0(\lambda_{inm}, m, n) = \int_0^1 \left[\dot{U}_{0c}^* K_1 + \dot{V}_{0s}^* K_2 + \dot{W}_{0c}^* K_3 \right] \cdot r dr$$

Investigating the system (2.12) we can come across two cases, i.e. $m = 0$ and $m \neq 0$. When $m = 0$ an axisymmetric problem is considered, the solution of which has been obtained by the author and is described in the work [10].

To solve (2.12) when $m \neq 0$ new functions K_5, K_6 are introduced on the basis of the following representations:

$$K_5 = r^s K_1, \quad K_6 = r^s K_2 \qquad (s = \pm 1) \tag{2.14}$$

Then the particular solutions of the system of differential equations are found by the method of decomposition of functions $K_3 \div K_6$ into the following power series:

$$\{K_3, K_4, K_5, K_6\} = r^{\beta} \sum_{f=0,2,4}^{\infty} \{E_f, R_f, Y_f, P_f\} r^f \quad (\beta = \text{const}). \tag{2.15}$$

After substitution (2.15) in (2.12) we equate all multipliers with the same degree to zero and obtain values for the parameter β, as well as expressions for the coefficients E_f, R_f, Y_f, P_f. The result is four partial solutions which allow representing functions $K_1 \div K_4$.

Substituting functions $K_1 \div K_4$ in the boundary conditions at $r = 1$ (2.13) forms a homogeneous system of equations for constants $D_1 \div D_4$. Seeking for its non-trivial solution, we obtain a transcendental equation for computing λ_{inm} eigenvalues and expressions for $D_1 \div D_4$. Thus obtained solutions also meet the conditions of regularity of the solution in the center of the plate (the boundary conditions (2.13) at $r = 0$). Consistently applying inversion formulas (2.9), (2.3), (2.2) to the transformant (2.11) we obtain, taking into account (2.1) and (2.7), the following decomposition for $U(r,\theta,z,t), V(r,\theta,z,t), W(r,\theta,z,t), \varphi(r,\theta,z,t)$:

$$U(r,\theta,z,t) = H_1 + \frac{2}{L}\sum_{n=1}^{\infty}\left\{\sum_{m=0}^{\infty} P_m^{-1}\left[H_4 + \sum_{i=1}^{\infty} GK_1 \|K_{inm}\|^{-2}\right]\cos(m\theta)\right\}\sin j_n z, \tag{2 .16}$$

$$V(r,\theta,z,t) = H_1 + \frac{2}{L}\sum_{n=1}^{\infty}\left\{\pi^{-1}\sum_{m=1}^{\infty}\left[H_5 + \sum_{i=1}^{\infty} GK_2 \|K_{inm}\|^{-2}\right]\sin(m\theta)\right\}\sin j_n z,$$

$$W(r,\theta,z,t) = H_2 + \sum_{n=0}^{\infty}\Omega_n^{-1}\left\{\sum_{m=0}^{\infty} P_m^{-1}\left[H_6 + \sum_{i=1}^{\infty} GK_3 \|K_{inm}\|^{-2}\right]\cos(m\theta)\right\}\cos j_n z,$$

$$\varphi(r,\theta,z,t) = H_3 + \sum_{n=0}^{\infty}\Omega_n^{-1}\left\{\sum_{m=0}^{\infty} P_m^{-1}\left[H_7 + \sum_{i=1}^{\infty} GK_4 \|K_{inm}\|^{-2}\right]\cos(m\theta)\right\}\cos j_n z.$$

The potential difference $Q(t_*)$ between the end surfaces with electrodes in the piezoceramic cylinder is determined by the following equality:

$$Q(t_*) = (\pi a^2)^{-1} \int_0^{2\pi}\int_0^{a} \left[\varphi(r_*,\theta,L,t_*) - \varphi(r_*,\theta,0,t_*)\right]\cdot r_* dr_* d\theta. \tag{2.17}$$

Numerical analysis of results. As an example, we consider piezoceramic cylinders made from ceramic compositions PZT-4, PZT-19, which have the following physical characteristics of the material:

PZT–4: $\{e_{31}, e_{33}, e_{15}\} = \{-5.2,\ 15.1,\ 12.7\}$ Кл/м^2, $\{\varepsilon_{11}, \varepsilon_{33}\} = \{6.46,\ 5.62\} \times 10^{-9}$ Ф/м, $\{C_{11}, C_{12}, C_{13}, C_{33}, C_{55}, C_{66}\} = \{13.9,\ 7.8,\ 7.4,\ 11.5,\ 2.5,\ 3.0\} \times 10^{10}$ Н/м^2, $\rho = 7700$ кг/м^3, $\{k_p, k_{15}, k_{33}\}^* = \{0.58,\ 0.39,\ 0.6\}$;

PZT-19: $\{e_{31}, e_{33}, e_{15}\} = \{-4.9,\ 14.9,\ 10.6\}$ Кл/м^2, $\{\varepsilon_{11}, \varepsilon_{33}\} = \{7.73,\ 7.26\}\times 10^{-9}$ Ф/ м, $\{C_{11}, C_{12}, C_{13}, C_{33}, C_{55}, C_{66}\} = \{10.9,\ 6.1,\ 5.4,\ 9.3,\ 2.4,\ 2.4\}\times 10^{10}$ Н/м^2, $\rho = 7730$ кг/м^3, $\{k_p, k_{15}, k_{33}\} = \{0.56,\ 0.29,\ 0.64\}$ [2].

The table shows the numerical values of the spectrum of natural frequencies ω_{inm} $(m = 0,1,2)$; piezoceramic (PZT-4, PZT-19 – respectively upper and averages) and the ceramic cylinder (the bottom number) element having elastic characteristics similar to the composition of the PZT-19.

The numerical values of the elastic characteristics of piezoelectric ceramics PZT-4 more than the same value was PZT-19, at about the same electrical parameters. This results in a higher range of frequencies of natural oscillations. The highest difference is 9%. In addition, the first natural frequency of non-axis- symmetric oscillations corresponds to the formation of one half-wave on the angular coordinate ($m = 1$) and along the cylindrical surface $(n = 1)$ of the element under investigation.

In addition, connectivity electroelastic fields having dimensions of the cylinder has a significant impact on the entire frequency spectrum and leads to an increase in the numerical values. The greatest difference between the calculated for the piezoceramic (averages number) and elastic (the bottom number) elements, reaches 8.5%.

Table

ω_{inm} кГц	$m=0$			$m=1$			$m=2$		
	$n=1$	$n=2$	$n=3$	$n=1$	$n=2$	$n=3$	$n=1$	$n=2$	$n=3$
$i=1$	122.4 113.4 98.3	276.5 257.2 166.7	336.1 317.9 246.2	80.6 77.2 77.0	173.1 165.5 163.8	262.4 251.6 246.6	100.8 92.8 92.7	179.2 169.6 169.2	265.8 252.5 250.2
$i=2$	206.9 187.2 152.0	377.3 344.3 227.5	426.7 396.5 297.6	126.5 117.5 111.9	200.2 190.6 184.7	281.3 273.1 267.1	158.9 145.2 133.5	221.8 209.2 199.4	298.9 285.5 276.8
$i=3$	245.6 226.9 200.3	444.3 434.5 285.7	487.5 485.1 351.4	173.8 158.7 136.3	227.2 214.1 203.2	299.6 287.5 281.6	213.4 195.1 170.3	265.8 247.1 230.0	326.6 315.7 300.4

Fig.1 shows graphs of the variation of vertical displacements $W(r,\theta,z,t)$ and the potential difference in time $Q(t)$ when evenly distributed harmonic load of q_0 intensity and with a frequency of forced oscillations β acts on the end surface of the cylinder composition PZT-19 ($0 \le r \le 1,\ 0 \le \theta \le \pi$): $q_1(r,\theta,t) = q_1(r,\theta,t) = q_0 H(\pi - \theta)\sin\beta t$, where $H(...)$ – is the Heaviside unit function.

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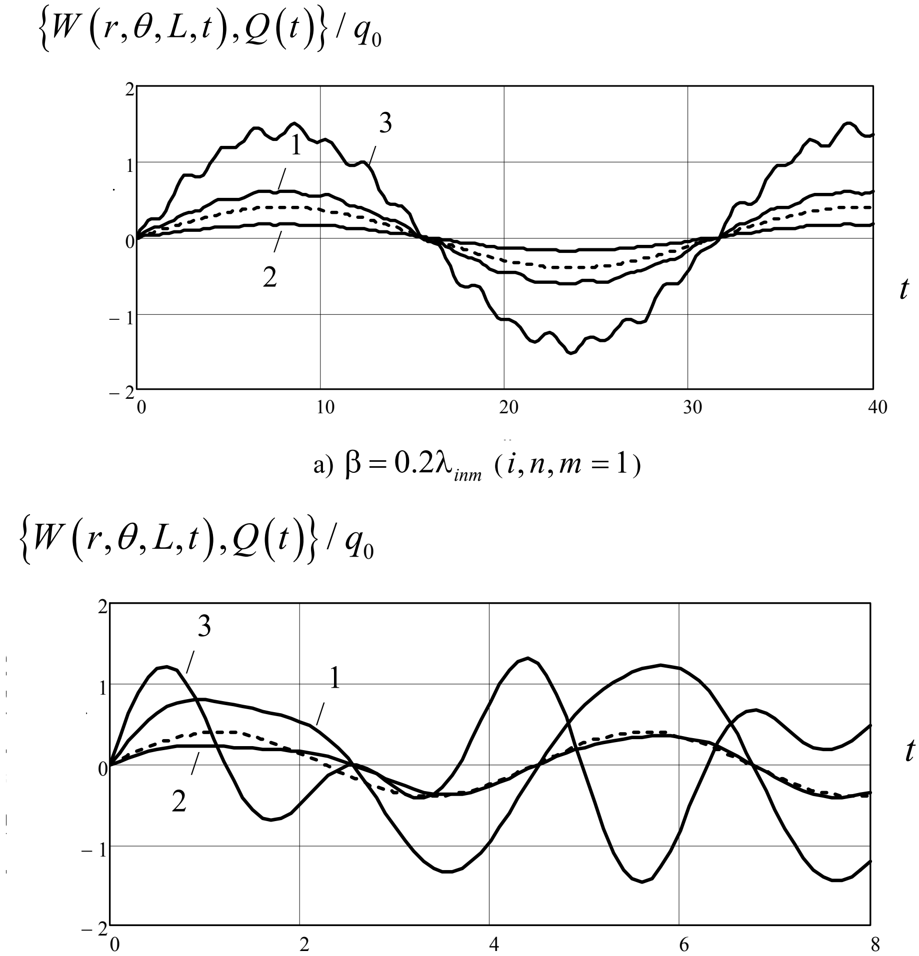
[2] Electromechanical coupling coefficients: k_p, k_{15}, k_{33} – respectively planar, shear, tension (compression) in thickness.



a) $\beta = 0.2\lambda_{inm}$ $(i,n,m=1)$

b) $\beta = 0.7\lambda_{inm}$ $(i,n,m=1)$

Fig.1. Graphsof $W(r,\theta,z,t)$ and $Q(t)$ in time:

$$1-W(1,\pi/2,0,t),\ 2-W(1,3\pi/2,0,t),\ 3-Q(t)$$

Functions $W(1,\pi/2,0,t)$, $W(1,3\pi/2,0,t)$, $Q(t)$ are denoted by figures 1, 2, 3, respectively and the dotted line shows the nature of the changes in the external load with time.

It is obvious that the vertical component of the vector of displacements at the non-loaded section at $\theta = 3\pi/2$ is significantly less than the corresponding values in the zone of the load action at $\theta = \pi/2$.

The calculation results also show that under harmonic loads the assumption of steady state of forced vibrations used in the study of dynamic problems is true only when frequencies of forced oscillations are substantially less than the first natural frequency. At the high – frequency external influence due to the superposition of the reflected waves of deformation there is a more complex dependence of the change in the stress-strain state and the electric field of the system in time.

Conclusions. On the basis of the conducted researches it is possible to formulate the main results:

1) There has been built a new closed solution using the basic calculation ratios to describe the operation of the typical elements of piezoceramic transducers of resonant and non-resonant classes in the form of a solid cylinder exposed to dynamic non-axis-symmetric mechanical loads. In particular the design of piezoelectric power generators [11];
2) Numerical calculation results show that the use of the constructed algorithm of calculation allows in comparison with numerical methods to obtain more accurate values of the range of natural frequencies, the stress-strain state and the electric field of the piezoceramic cylinder;
3) In the case of high-frequency external harmonic load the assumption of steady-state forced oscillations cannot be used in the study of elastic and electro-elastic systems;
4) Calculated ratios allow you to automate the research, which significantly increases the theoretical level of engineering calculations.

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