ՀԱՅԱՍՏԱՆԻ ԳԻՏՈՒԹՅՈՒՆՆԵՐԻ ԱԶԳԱՅԻՆ ԱԿԱԴԵՄԻԱՅԻ ՏԵՂԵԿԱԳԻՐ
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ORTHOGONALITY RELATIONSHIP FOR SHEAR WAVE MODES IN PIEZOELECTRIC WAVEGUIDE

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Keywords: waveguide, piezoelectric, shear waves, orthogonality
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Условие ортогональности для мод сдвиговых волн в пьезоэлектрическом волноводе

В рамках полной системы уравнений элетродинамики Максвелла и динамических уравнений теории упругости получено условие ортогональности для мод связанных электромагнитоупругих сдвиговых волн в пьезоэлектрическом волноводе.

Ղազարյան Կ.Բ

Պյեզոէլեկտրիկ ալիքատարում սահքի ալիքների մոդաների օրթոգոնալության պայմանը

Մաքսվելի էլեկտրադինամիկայի հավասարումների լրիվ համակարգի և առաձգականության տեսության դինամիկ գծային հավասարումների հիման վրա դիտարկված են էլեկտրաառաձգական սահքի ալիքների տարածումը և ստացված է ալիքների մոդաների օրթոգոնալության պայմանը պյեզոէլեկտրիկ ալիքատարում

In the framework of the full set of Maxwell equations and dynamic equations of the elasticity theory an orthogonality relationship is derived for interconnected electro-magneto-elastic shear wave modes in piezoelectric waveguide.

To tackle the problems of electromagnetic and elastic wave propagation in waveguides with complicated geometrical and physical properties a modal decomposition approach is used based on the eigen function expansion of the electromagnetic fields vectors, elastic displacements and stresses, where the considered eigen functions are wave modes in an infinite homogeneous waveguide. The eigen function orthogonality relation between modes for the Rayleigh–Lamb waves at constant frequency but differing wavenumbers was first derived in [1]. The completeness for Rayleigh–Lamb modes in homogeneous waveguides is proved in [2], where orthogonality and biorthogonality relations are given also. Based on the quasi-static approximation of Maxwell equations the orthogonality condition is derived in [3] for piezoelectric crystal displacement vibration modes associated with different eigen-frequencies.

Based on the modal decomposition method the Rayleigh–Lamb wave propagation in inhomogeneous, variable thickness waveguides are studied in [4,5]. This method was also used for Rayleigh-Lamb waves in a waveguide with a periodic structure [6].

In the framework of the full set of Maxwell equations and dynamic equations of the elasticity theory the propagation of SH waves in piezoelectric crystals is considered in [7-11]. For waves in waveguides with electrically shorted (electroted) and open surfaces exact solutions are obtained in [9]. Electro-magneto-elastic wave propagation oblique to the periodic phononic/photonic structure is discussed in [10-11].

We are interested in the propagation of electro-magneto-elastic shear coupled waves in an infinite piezoelectric waveguide. Waveguide bounded by plane surfaces $y = \pm h/2$, so that the plane x coincides with the middle surface of the waveguide, the axis z direction coincides with the direction of the hexagonal crystal crystallographic axis.

In the framework of the full set of Maxwell equations and dynamic equations of the elasticity theory, the interconnected elastic and electric excitations in piezoelectric crystal decouple into plane and anti-plane problems. The anti-plane problem is described by the following equations and relations defining SH waves in a waveguide [8].

$$\frac{\partial\sigma_{xz}}{\partial x}+\frac{\partial\sigma_{yz}}{\partial y}=\rho\frac{\partial^2 u_z}{\partial t^2},\quad \frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}=-\mu_{33}\frac{\partial H_z}{\partial t},$$

$$\frac{\partial H_z}{\partial y}=\frac{\partial D_x}{\partial t},\quad -\frac{\partial H_z}{\partial x}=\frac{\partial D_y}{\partial t}, \tag{1}$$

$$D_x=e_{15}\frac{\partial u_z}{\partial x}+\varepsilon_{11}E_x,\quad D_y=e_{15}\frac{\partial u_z}{\partial y}+\varepsilon_{11}E_y,$$

$$\sigma_{xz}=c_{44}\frac{\partial u_z}{\partial x}-e_{15}E_x,\quad \sigma_{yz}=c_{44}\frac{\partial u_z}{\partial y}-e_{15}E_y$$

Here u_z is component of the displacement vector, σ_{xz},σ_{yz} are the stress tensor components, H_z is the component of the magnetic field vector, E_x,E_y are components of the electric field vector, c_{44}, is the elastic stiffness coefficient, D_x,D_y are the components of electrical displacement vector, ε_{11} is the electrical permittivity coefficient, e_{15} is the piezoelectric stress coefficient, μ_{33} is the magnetic permeability, ρ is the bulk density. Harmonic time dependence, $\exp(i\omega t)$ for all physical variables with ω as wave angular frequency is assumed henceforth.

We take the following notations

$$e_{15}=e,\ c_{44}=G,\ \varepsilon_{11}=\varepsilon,\ H=i\omega H_z,\ \sigma_{xz}=\sigma,\ E_y=E,\ u_z=U$$

In order to derive an orthogonality relation it will be convenient to work on a set of four functions $U(x,y),\ E(x,y),\ H(x,y),\ \sigma(x,y)$. Excluding functions D_x,D_y from equations (1) after some transformations we come to the following simultaneous equations

$$\frac{\partial E}{\partial x}=\left(\frac{G}{G_0\varepsilon}\frac{\partial^2}{\partial y^2}+\mu\omega^2\right)H-\frac{e}{G_0\varepsilon}\frac{\partial\sigma}{\partial y};\qquad \frac{\partial U}{\partial x}=\frac{e}{G_0\varepsilon}\frac{\partial H}{\partial y}+\frac{\sigma}{G_0}; \tag{2}$$

$$\frac{\partial H}{\partial x}=-\varepsilon E-e\frac{\partial U}{\partial y};\qquad \frac{\partial\sigma}{\partial x}=e\frac{\partial E}{\partial y}-\left(G\frac{\partial^2}{\partial y^2}+\rho\omega^2\right)U;$$

where $G_0 = G + e^2/\varepsilon$;

The functions $\sigma_{yz}(x,y)$, $E_x(x,y)$ via functions $U(x,y), \sigma(x,y), H(x,y), E(x,y)$ can be expressed as

$$\sigma_{yz} = G\frac{\partial U}{\partial y} - eE, \qquad E_x = \frac{G}{G_0\varepsilon}\frac{\partial H}{\partial y} - \frac{e\sigma}{G_0\varepsilon}; \tag{3}$$

Introducing vectors $\vec{Z} = (E,U)^T$; $\vec{R} = (H,\sigma)^T$ ~ we can rewrite the set of equations (2) in a matrix operator form

$$\frac{\partial \vec{Z}}{\partial x} = \widehat{F}\vec{R}; \qquad \frac{\partial \vec{R}}{\partial x} = \widehat{Q}\vec{Z}$$

or

$$\frac{\partial}{\partial x}\begin{pmatrix}\vec{Z}\\ \vec{R}\end{pmatrix} = \begin{pmatrix}0 & \widehat{F}\\ \widehat{Q} & 0\end{pmatrix}\begin{pmatrix}\vec{Z}\\ \vec{R}\end{pmatrix} \tag{4}$$

where $\widehat{F}, \widehat{Q}$ are the operator matrices

$$\widehat{F} = \begin{pmatrix} \frac{G}{G_0\varepsilon}\partial_{yy} + \mu\omega^2 & -\frac{e}{G_0\varepsilon}\partial_y \\ \frac{e}{G_0\varepsilon}\partial_y & \frac{1}{G_0} \end{pmatrix}; \qquad \widehat{Q} = \begin{pmatrix} -\varepsilon & -e\partial_y \\ e\partial_y & -(G\partial_{yy} + \rho\omega^2) \end{pmatrix};$$

Let us define the inner product of two vectors as

$$\left\langle \vec{f} \,\middle|\, \vec{g} \right\rangle = \left\langle \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \middle| \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \right\rangle = \int_{-h/2}^{h/2} (f_1 g_1 + f_2 g_2)\, dy$$

Operators $\widehat{F}, \widehat{Q}$ have the following properties (see Appendix A)

$$\left\langle \widehat{F}\begin{pmatrix} H \\ \sigma \end{pmatrix} \middle| \begin{pmatrix} \tilde{H} \\ \tilde{\sigma} \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} H \\ \sigma \end{pmatrix} \middle| \widehat{F}\begin{pmatrix} \tilde{H} \\ \tilde{\sigma} \end{pmatrix} \right\rangle + \eta(h)$$

$$\left\langle \widehat{Q}\begin{pmatrix} E \\ U \end{pmatrix} \middle| \begin{pmatrix} \tilde{E} \\ \tilde{U} \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} E \\ U \end{pmatrix} \middle| \widehat{Q}\begin{pmatrix} \tilde{E} \\ \tilde{U} \end{pmatrix} \right\rangle + \xi(h) \quad (5)$$

$$\eta(h) = \left(H\tilde{E}_x - \tilde{H}E_x\right)\Big|_{y=-h/2}^{y=h/2}; \qquad \xi(h) = \left(\tilde{\sigma}_{yz}U - \sigma_{yz}\tilde{U}\right)\Big|_{y=-h/2}^{y=h/2}$$

Here vectors $(E,U)^T, (H,\sigma)^T, (\tilde{E},\tilde{U})^T, (\widehat{H},\tilde{\sigma})^T$ are linear independent solutions of the matrix equation (4).

Thus, for all cases where the terms outside the integrals having vanished because of boundary conditions we get that operators $\widehat{F},\widehat{Q}$ are formally self-adjoint operators.

When $\eta(h) = 0;\ \xi(h) = 0$, using the properties (5) we get

$$\left\langle \widehat{F}\widehat{Q}\begin{pmatrix} E \\ U \end{pmatrix} \middle| \begin{pmatrix} \tilde{H} \\ \tilde{\sigma} \end{pmatrix} \right\rangle = \left\langle \widehat{Q}\begin{pmatrix} E \\ U \end{pmatrix} \middle| \widehat{F}\begin{pmatrix} \tilde{H} \\ \tilde{\sigma} \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} E \\ U \end{pmatrix} \middle| \widehat{Q}\widehat{F}\begin{pmatrix} \tilde{H} \\ \tilde{\sigma} \end{pmatrix} \right\rangle \quad (6)$$

Now we expand all functions in the form

$$f(x,y) = \sum_n f^{(n)}(y)\exp(-ik_n x) \quad (7)$$

where wavenumbers k_n are eigenvalues, $f^{(n)}(y)$ are eigen functions.

Introducing different modes of eigen functions $(\overrightarrow{Z_n}, \overrightarrow{R_n})^T, \vec{Z}_n = (E_n, U_n)^T, \vec{R} = (H_n, \sigma_n)^T$, we come to the self-adjoint eigenvalue problem

$$ik_n\begin{pmatrix} \overrightarrow{Z_n} \\ \overrightarrow{R_n} \end{pmatrix} = \begin{pmatrix} 0 & \widehat{F} \\ \widehat{Q} & 0 \end{pmatrix}\begin{pmatrix} \overrightarrow{Z_n} \\ \overrightarrow{R_n} \end{pmatrix} \quad (8)$$

with boundary conditions

$$\eta(h) = \left(H_n E_{xm} - H_m E_{xn}\right)\Big|_{y=-h/2}^{y=h/2} = 0; \qquad \xi(h) = \left(\sigma_{yzn}U_m - \sigma_{yzm}U_n\right)\Big|_{y=-h/2}^{y=h/2} = 0$$

Now our important point is to derive an othogonality relation between different modes of the eigen functions.

From (8) it follows that vectors $\overrightarrow{Z_m}, \overrightarrow{R_n}$ associated with different modes satisfy the equations

$$\widehat{F}\widehat{Q}\overrightarrow{Z_m} = -k_m^2\overrightarrow{Z_m}; \qquad \widehat{Q}\widehat{F}\overrightarrow{R_n} = -k_n^2\overrightarrow{R_n}$$

Multiplying the first equation on $\overrightarrow{R_n}$, the second on $\overrightarrow{Z_m}$ and taking inner products of these vectors, we come to the following relation

$$\left(k_m^2 - k_m^2\right)\left\langle\overrightarrow{Z_m}\middle|\overrightarrow{R_n}\right\rangle = \left\langle\widehat{F}\widehat{Q}\overrightarrow{Z_m}\middle|\overrightarrow{R_n}\right\rangle - \left\langle\overrightarrow{Z_m}\middle|\widehat{Q}\widehat{F}\overrightarrow{R_n}\right\rangle$$

When $\eta\left(h\right) = 0;\ \xi\left(h\right) = 0$ taking into account (6) we get the following orthogonality relationship for eigen-functions associated with wavenumber eigenvalues k_n

$$\int_{-h}^{h}\left(U_n\sigma_m + E_nH_m\right)dy = 0 \qquad n \neq m \tag{9}$$

This relationship is similar to the orthogonality relationship for the Rayleigh-Lamb wave modes in traction free waveguide [2]

$$\int_{-h}^{h}\left(U_x^{(n)}\sigma_{xx}^{(m)} - \sigma_{xy}^{(n)}U_y^{(m)}\right)dy = 0 \quad n \neq m \tag{10}$$

In [3] based on quasi-static approximation of Maxwell equations the other type of orthogonality relation was derived for piezoelectric displacement vibration modes associated with eigen-frequencies ω_n .

This orthogonality relation for bounded piezoelectric body of volume V with traction free and electrically shorted surface has the following form [3],

$$\iiint_V \vec{U}_n\vec{U}_m dv = 0 \qquad m \neq n$$

where $\vec{U}_m$ are elastic displacement vibration modes associates with piezoelectric body eigen-frequencies ω_m .

Note the following sets of boundary conditions ensuring orthogonality relationship (9)

a) Traction free and electrically shorted wall b) clamped and magnetically closed wall, c) clamped and electrically shorted wall, d) traction free and electrically open wall.

Now we will bring the example of orthogonality relationship for waveguide with non-symmetrical boundary conditions at waveguide walls (plane x coincides with the a wall of the waveguide)

$$y = 0; \qquad U = 0; \qquad E_x = 0$$
$$y = h \qquad \sigma_{yz} = 0; \qquad E_x = 0$$

Solutions of equations (4) can be derived as

$$E(x,y) = \sum_n A_n(x) E_n(y); \qquad U(x,y) = \sum_n A_n(x) U_n(y)$$
$$H(x,y) = \sum_n B_n(x) H_n(y); \qquad \sigma(x,y) = \sum_n B_n(x) \sigma_n(y)$$

where $E_n(y), U_n(y), H_n(y), \sigma_n(y)$ are the following eigenfunctions

$$E_n(y) = -\frac{e\left(q_n r_n \cos(q_n y) - k_n^2 \gamma_n \cos(r_n y)\right)}{r_n \varepsilon}; \qquad U_n(y) = \sin(q_n y);$$
$$H_n(y) = \frac{\mathrm{i} k_n \gamma_n e}{r_n} \cos(r_n y); \qquad \sigma_n(y) = \mathrm{i} k_n \left(\frac{e^2 \gamma_n}{\varepsilon} \sin(r_n y) + G_0 \sin(q_n y) \right)$$

$$A_n(x) = A_n \exp(ik_n x) + B_n \exp(-ik_n x); \qquad B_n(x) = A_n \exp(ik_n x) - B_n \exp(-ik_n x);$$
$$r_n = \sqrt{\varepsilon\mu\omega^2 - k_n^2}; \qquad q_n = \sqrt{\frac{\rho}{G_0}\omega^2 - k_n^2}; \qquad \gamma_n = -\frac{\sin(q_n h)}{\sin(r_n h)};$$

The positive and negative eigen wavenumbers k_n are determined as roots of the following equation

$$\frac{e^2 k_n^2}{G_0 q_n r_n \varepsilon} \cos(r_n h) \sin(h q_n) + \sin(r_n h) \cos(q_n h) = 0$$

The functions $\sigma_{\mathrm{yz}}, E_x$ can be defined as

$$E_{xn} = -iek_n \left(\gamma \sin(r_n y) + \sin(q_n y) \right)$$
$$\sigma_{yzn} = G_0 q_n \left(\cos(q_n y) - \frac{e^2 k_n^2 \gamma}{\varepsilon r_n q_n G_0} \cos(r_n y) \right)$$

The orthogonality relationship for eigenfunctions can be written as

$$\int_0^h (U_n \sigma_m + E_n H_m) dy = J_n \delta_{nm}$$

where δ_{nm} is the Kronecker symbol,

$$J_n = \frac{\mathrm{i}k_n}{2q_n r_n^3}\left\{ hq_n r_n^3 + \frac{q_n e^2}{\varepsilon G_0}\left[(k_n^2 + 2r_n^2)\cot(hr) + hk_n^2 r_n \csc^2(hr_n) \right] \times \right.$$
$$\left. \times \sin^2(hq_n) - r_n^3 \sin(hq_n)\cos(hq_n) \right\} \tag{11}$$

Conclusion

For shear coupled wave the orthogonality relationship is derived in a waveguide between different modes of the eigen functions (elastic displacement, stress, electrical and magnetic field intensities), associated with the wavenumber eigenvalues. For waveguide with non-symmetrical boundary conditions at waveguide walls the dispersion equation determining Eigen wavenumbers are obtained and the corresponding orthogonality relationship is derived.
The derived relationships can be helpful to tackle a number problems of wave prorogation in a photon-phonon periodic piezoelectric waveguide, in determination of solutions in reflection, transmission problems in inhomogeneous, layered piezoelectric waveguides with different boundary conditions at waveguide walls.

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Appendix A

Properties of matrices $\widehat{F}$ and $\widehat{Q}$

Let $\vec{S} = (s_1, s_2)^T ; \vec{P} = (p_1, p_2)^T$ are arbitrary linear independent vectors.
From (4) it follows that

$$\left\langle \widehat{F}\vec{S} \middle| \vec{P} \right\rangle = \left\langle \vec{S} \middle| \widehat{F}\vec{P} \right\rangle + \frac{1}{\varepsilon G_0}\left[G\left(p_1 \partial_y s_1 - s_1 \partial_y p_1 \right) - e\left(s_1 p_2 - p_1 s_2 \right) \right]\Big|_{y=-h/2}^{y=h/2};$$

$$\left\langle \widehat{Q}\vec{S} \middle| \vec{P} \right\rangle = \left\langle \vec{S} \middle| \widehat{Q}\vec{P} \right\rangle + \left[e\left(s_1 p_2 - p_1 s_2 \right) + G\left(s_2 \partial_y p_2 - p_2 \partial_y s_2 \right) \right]\Big|_{y=-h/2}^{y=h/2};$$

$$\left\langle \widehat{F}\widehat{Q}\vec{S} \middle| \vec{P} \right\rangle = \left\langle \widehat{Q}\vec{S} \middle| \widehat{F}\vec{P} \right\rangle +$$
$$+ \frac{1}{\varepsilon G_0}\left[G\left(p_1 \partial_y \left(\widehat{Q}\vec{S}\right)_1 - \left(\widehat{Q}\vec{S}\right)_1 \partial_y p_1 \right) - e\left(\left(\widehat{Q}\vec{S}\right)_1 p_2 - p_1 \left(\widehat{Q}\vec{S}\right)_2 \right) \right]\Big|_{y=-h/2}^{y=h/2} =$$

$$= \left\langle \vec{S} \middle| \widehat{Q}\widehat{F}\vec{P} \right\rangle + \left[e\left(s_1 \left(\widehat{F}\vec{P}\right)_2 - \left(\widehat{F}\vec{P}\right)_1 s_2 \right) + G\left(s_2 \partial_y \left(\widehat{F}\vec{P}\right)_2 - \left(\widehat{F}\vec{P}\right)_2 \partial_y s_2 \right) \right]\Big|_{y=-h/2}^{y=h/2} +$$
$$+ \frac{1}{\varepsilon G_0}\left[G\left(p_1 \partial_y \left(\widehat{Q}\vec{S}\right)_1 - \left(\widehat{Q}\vec{S}\right)_1 \partial_y p_1 \right) - e\left(\left(\widehat{Q}\vec{S}\right)_1 p_2 - p_1 \left(\widehat{Q}\vec{S}\right)_2 \right) \right]\Big|_{y=-h/2}^{y=h/2}$$

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