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ИЗВЕСТИЯ НАЦИОНАЛЬНОЙ АКАДЕМИИ НАУК АРМЕНИИ

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IN-PLANE INVERSE PROBLEM ON CRACK IDENTIFICATION IN THE ELASTIC HALF-SPACE

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Ключевые слова: упругое полупространство, трещина, обратная задача, идентификация, интегральные уравнения, функционал невязки.
Key words: elastic half-space, crack, inverse problem, identification, integral equations, discrepancy functional.

Չարլետտա Մ., Իովանե Ջ., Սումբատյան Մ.Ա.
Առաձգական կիսատարածությունում ճաքի նույնականացման հարթ հակադարձ խնդիրը
Աշխատանքում դիտարկվում է հարթ խնդիր համասեռ և իզոտրոպ առաձգական կիսատարածության համար: Աշխատանքի նպատակն է կիսատարածության ներսում ճաքի վերականգնման հետ կապված հակադարձ խնդրի լուծման համար արդյունավետ մաթեմատիկական ապարատի մշակումը: Մասնավորապես, որոշվում է եզրային մակերևույթին զուգահեռ տեղակայված ուղղագիծ ճաքի դիրքը և չափսերը: Դիտարկվող հակադարձ խնդրի ձևակերպումը հիմնված է առաջին սեռի ինտեգրալ հավասարումների համակարգի վրա:

Чарлетта М., Иоване Дж., Сумбатян М.А.
Плоская обратная задача идентификации трещины в упругом полупространстве
В работе рассматривается плоская задача для упругого однородного и изотропного полупространства. Целью работы является разработка эффективного математического аппарата для решения обратной задачи, которая связана с реконструкцией трещины внутри полупространства. В частности, определяется положение и размер прямолинейной трещины, параллельной её граничной поверхности. Формулировка рассматриваемой обратной задачи основана на системе интегральных уравнений первого рода.

In this work we study a homogeneous and isotropic elastic half-space in the context of in-plane deformation. The aim of the paper is to propose a powerful mathematical tool to solve the inverse problem, which is connected to crack reconstruction inside the half-space. In particular, position and sizes of the linear crack, parallel to its boundary surface, are determined. The formulation of the considered inverse problem is based on a system of integral equations of the first kind.

1. Introduction

The theory of inverse problems is an intensively developing branch of applied mathematics and engineering science. The current state of the art and further references can be found, for example, in [1-3].

The recovery of a linear crack by boundary measurements is an extensively studied problem. It is known that in geomechanics and strength analysis one needs to reconstruct geometry (i.e. position, shape, and characteristic size) of linear cracks from results of the measurements of some physical fields over boundary surface of the considered elastic solid. This problem seems to be a typical inverse problem.

Various methods were applied to study the inverse problem on reconstruction of crack's geometry. The concept of duality based on the reciprocity gap principle gives an efficient instrument to study such a type of problems in some cases. Among other important works published in this field we can mention here [4], where the authors establish uniqueness of the problem under some overdetermined boundary measurements, that is to know on the (total) boundary surface or line, in 2-d case, both the value of the basic potential function and the value of its normal derivative. These results were advanced in [5,6] where it is shown that to find the normal to the plane of the crack is a more easy task than to determine its true configuration. We would only stress that these results require the input data measured over full boundary surface. Some interesting ideas for the case of incomplete data were proposed in [7].

In the problem under investigation it is quite natural to operate with the settlement of the boundary surface, which typically can be measured with a high precision. For this purpose, we may apply some outer loads to the boundary surface. Then its settlement depends on geometry and position of a system of discontinuities, located inside the medium. Hence, we can pose the inverse problem on reconstruction of these geometric parameters from the input data taken from results of the measurements.

In the present paper we consider a homogeneous and isotropic elastic half-space in the context of in-plane deformation, whose formulation is more complex when compared with the case of scalar model. We study the problem connected with the reconstruction of the position and the size of a linear crack inside an elastic half-space and parallel to its boundary surface. This study, which defines a typical inverse problem, can be reduced to a system of integral equations of the first kind. It is known that such equations belong to a class of the so-called ‘‘ill-posed’’ problems [2]. This means that application of ordinary numerical approaches to such problems makes calculations unstable. In order to overcome this difficulty, one should apply some refined methods like Tikhonov's smoothing functional, regularizing operators, or similar. In the context of mathematical formulation of the problem we notice that it is simultaneously ill-posed and non linear problem, like many other inverse problems (see [2]).

Another relevant aspect of the inverse problem concerned is a continuity of the solution upon the input data, as well as the smoothness and the stability of the proposed algorithms. A good survey on this subject is given in [8], where the reader can find also some results concerning Lipschitz stability.

A motivation why it is important to reconstruct cracks parallel to the free surface of the elastic half-space can be justified as follows. First of all, before to study a general case of the crack location, it is quite natural to study the simpler case of parallel crack. In fact, with the use of the Fourier transform in this problem all kernels of respective integral equations are expressed in terms of elementary functions, as well as their right-hand sides. Besides, this geometry is very important for applications in the testing of some composite materials on epoxy basis. Really, in the layered composites produced sequentially from layers of reinforcing strings and layers of epoxy, the layerwise process, there may appear exfoliations between contacting layers, which are obviously parallel to the free surface.

Similar problems in the less complex anti-plane case, both for horizontal and inclined cracks, have been studied by the authors in [9,10].

2. Mathematical Formulation and Reducing to Integral Equations

Let us consider the in-plane problem concerning a linear horizontal crack located in the homogeneous and isotropic elastic half-plane parallel to its boundary surface (see Fig.1).

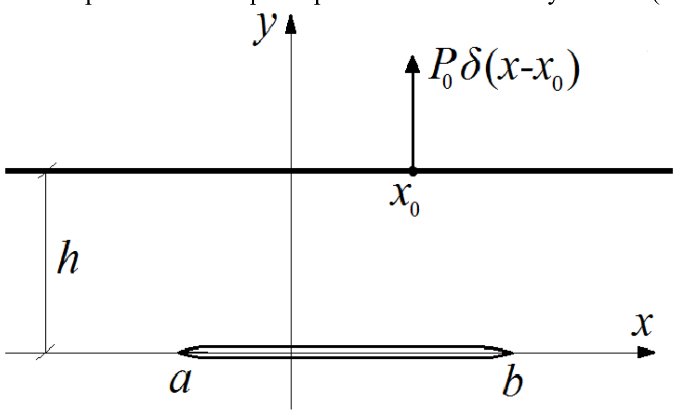

Fig. 1

In the considered two-dimensional (2D) case of plane strain the formulation of the problem implies the components of the displacement vector $\bar{u}$ to be of the following form

$$\bar{u}(x,y,z)=\{u_x(x,y),u_y(x,y),0\} \qquad (2.1)$$

where u_x and u_y – components of the displacement vector in the direction of x and y axes, respectively. In the studied 2D problem these two functions satisfy the following equations of equilibrium:

$$\frac{\partial^2 u_x}{\partial x^2}+c^2\frac{\partial^2 u_x}{\partial y^2}+(1-c^2)\frac{\partial^2 u_y}{\partial x\partial y}=0, \qquad \left(c^2=\frac{\mu}{\lambda+2\mu}<1\right)$$
$$\frac{\partial^2 u_y}{\partial y^2}+c^2\frac{\partial^2 u_y}{\partial x^2}+(1-c^2)\frac{\partial^2 u_x}{\partial x\partial y}=0 \qquad (2.2)$$

where λ and μ are elastic moduli.

If functions u_x and u_y are determined from Eq.(2.2) then the components of the stress tensor can be found from the constitutive equations

$$\frac{\sigma_{xx}}{\lambda+2\mu}=\frac{\partial u_x}{\partial x}+(1-2c^2)\frac{\partial u_y}{\partial y}, \qquad \frac{\sigma_{xy}}{\mu}=\frac{\partial u_x}{\partial y}+\frac{\partial u_y}{\partial x},$$
$$\frac{\sigma_{yy}}{\lambda+2\mu}=\frac{\partial u_y}{\partial y}+(1-2c^2)\frac{\partial u_x}{\partial x} \qquad (2.3)$$

Let us assume that the crack is linear, horizontal, and h designates the distance between the crack's line $y=0$ and the boundary surface $y=h$. Let us also assume that the left and the right tips of the crack have the Cartesian coordinates $(a,0)$ and $(b,0)$, respectively (see Fig.1).

Let a (known) normal point load $P_0(x)\delta(x-x_0)$ be applied to the boundary surface of the half-space at point x_0, which is assumed to be known. Then the full mathematical formulation of the direct problem is to solve equations (2.2) with the following boundary conditions

$$y=h: \quad \sigma_{xy}(x,h)=0, \quad \sigma_{yy}(x,h)=P_0\delta(x-x_0), \quad (|x|<\infty) \qquad (2.4a)$$

$$y=0:$$
$$\begin{cases} u_x(x,+0)=u_x(x,-0), \quad u_y(x,+0)=u_y(x,-0), & (x<a)\cup(x>b) \\ \sigma_{xy}(x,+0)=\sigma_{xy}(x,-0), \quad \sigma_{yy}(x,+0)=\sigma_{yy}(x,-0), & (x<a)\cup(x>b) \end{cases} \qquad (2.4b)$$

$$\sigma_{xy}(x,+0)=\sigma_{xy}(x,-0)=\sigma_{yy}(x,+0)=\sigma_{yy}(x,-0)=0, \quad (a<x<b) \qquad (2.4c)$$

In order to give a solution to the boundary value problem (2.2) – (2.4), we consider separately the upper layer $(|x|<\infty, 0<y<h)$, where all physical quantities are marked by the subscript "+", and the lower half-space $(|x|<\infty, y<0)$, where all quantities are marked by "–".

To the considered boundary problem, we construct the solution by applying the Fourier transform with respect to variable x, which for any given function $f(x,y)$ is defined by the pair of (direct and inverse) relations:

$$F(s,y)=\int_{-\infty}^{\infty} f(x,y)e^{isx}dx, \qquad f(x,y)=\frac{1}{2\pi}\int_{-\infty}^{\infty} F(s,y)e^{-isx}ds \tag{2.5}$$

Note that all Fourier transforms are designated by corresponding capital letters. It is evident that the first-order derivative $\partial/\partial x$ of any function in the Fourier variables will be replaced by factor $(-is)$, and the second-order derivative $\partial^2/\partial x^2$ – by factor $(-s^2)$. Then in Fourier images, system (2.2) becomes a system of ordinary differential equations, with a certain parameter s :

$$\begin{cases} c^2 U_x'' - s^2 U_x + (1-c^2)(-is)U_y' = 0 \\ (1-c^2)(-is)U_x' + U_y'' - c^2 s^2 U_y = 0 \end{cases} \tag{2.6}$$

where all ordinary derivatives are applied with respect to variable y .

Let us note that the Fourier images of the components of the stress tensor are expressed in terms of the Fourier transforms of functions u_x and u_y as follows:

$$\frac{\Sigma_{xx}}{\lambda+2\mu} = (-is)U_x + (1-2c^2)U_y', \qquad \frac{\Sigma_{xy}}{\mu} = U_x' + (-is)U_y,$$
$$\frac{\Sigma_{yy}}{\lambda+2\mu} = U_y' + (1-2c^2)(-is)U_x \tag{2.7}$$

The solution to system (2.6) can be constructed by using the method of characteristic polynomial that results in the following representations in the lower half-plane and in the upper strip, respectively:

$$\begin{cases} U_x^- = Ai\,\text{sign}(s)e^{|s|y} - B\left[\dfrac{1+c^2}{(1-c^2)(-is)} + iy\,\text{sign}(s)\right]e^{|s|y} \\ U_y^- = -Ae^{|s|y} + By\,e^{|s|y}, \qquad (y \le 0) \end{cases} \tag{2.8}$$

and

$$\begin{cases} U_x^+ = C\,i\,\text{ch}(sy) - D\left[\dfrac{1+c^2}{(1-c^2)(-is)}\text{ch}(sy) + iy\,\text{sh}(sy)\right] + \\ + Ei\,\text{sh}(sy) - F\left[\dfrac{1+c^2}{(1-c^2)(-is)}\text{sh}(sy) + iy\,\text{ch}(sy)\right] \\ U_y^+ = -C\,\text{sh}(sy) + Dy\,\text{ch}(sy) - E\,\text{ch}(sy) + Fy\,\text{sh}(sy), \qquad (0 \le y \le h) \end{cases} \tag{2.9}$$

where the six unknown constants A, B, C, D, E, F should be determined from boundary conditions (2.4).

On the basis of Eqs. (2.7) – (2.9) one can easily write out all physical quantities present in boundary conditions (2.4).

If we introduce the two new unknown functions $g_x(x), g_y(x)$, $a \le x \le b$, as follows

$$u_x^+(x,0)-u_x^-(x,0)=\begin{cases} g_x(x), & a<x<b \\ 0, & (x<a)\cup(x>b) \end{cases};$$
$$u_y^+(x,0)-u_y^-(x,0)=\begin{cases} g_y(x), & a<x<b \\ 0, & (x<a)\cup(x>b) \end{cases} \tag{2.10}$$

Then

$$U_x^+(s,0)-U_x^-(s,0)=G_x(s)=\int_a^b g_x(\xi)e^{is\xi}d\xi;$$
$$U_y^+(s,0)-U_y^-(s,0)=G_y(s)=\int_a^b g_y(\xi)e^{is\xi}d\xi \tag{2.11}$$

Obviously, the physical meaning of functions $g_x(x)$ and $g_y(x)$ is the relative displacement of the upper and the lower faces of the crack, in horizontal and vertical direction respectively.

Now, by using representations (2.10), (2.11), one can satisfy boundary conditions (2.4a), (2.4b) that results in a 6×6 linear algebraic system regarding coefficients A,B,C,D,E,F whose solution can easily be constructed in the following form:

$$A(s)=-iG_x e^{-2|s|h}\left\{\operatorname{sign}(s)[c^2\operatorname{sh}^2(\operatorname{sh})+(1-c^2)s^2h^2]+(c^2/2)[\operatorname{sh}(2\operatorname{sh})+2\operatorname{sh}]\right\}+$$
$$+G_y e^{-2|s|h}\left\{[\operatorname{sh}^2(\operatorname{sh})-(1-c^2)s^2h^2]+[\operatorname{sign}(s)/2][\operatorname{sh}(2\operatorname{sh})-2\operatorname{sh}]\right\}-\frac{P_0\,e^{isx_0}}{2\mu\, s(1-c^2)}e^{-2|s|h}\times \tag{2.12a}$$
$$\times\left\{\operatorname{sign}(s)[\operatorname{ch}(\operatorname{sh})+(1-c^2)\operatorname{sh}\ \operatorname{sh}(\operatorname{sh})]+[\operatorname{sh}(\operatorname{sh})+(1-c^2)\operatorname{sh}\ \operatorname{ch}(\operatorname{sh})]\right\}$$

$$B(s)=-is(1-c^2)G_x e^{-2|s|h}\left\{\operatorname{sh}^2(\operatorname{sh})+[\operatorname{sign}(s)/2][\operatorname{sh}(2\operatorname{sh})+2\operatorname{sh}]\right\}+$$
$$+s(1-c^2)G_y e^{-2|s|h}\left\{\operatorname{sign}(s)\operatorname{sh}^2(\operatorname{sh})+(1/2)[\operatorname{sh}(2\operatorname{sh})-2\operatorname{sh}]\right\}- \tag{2.12b}$$
$$-\frac{P_0\,e^{isx_0}}{2\mu}e^{-2|s|h}[\operatorname{ch}(\operatorname{sh})+\operatorname{sign}(s)\operatorname{sh}(\operatorname{sh})]$$

$$C(s)=A(s)+i\operatorname{sign}(s)c^2G_x,\qquad D(s)=B(s)+(1-c^2)isG_x,$$
$$E(s)=A(s)-G_y,\qquad F(s)=B(s)\operatorname{sign}(s)-(1-c^2)sG_y \tag{2.12c}$$

Now, the only remaining boundary condition is (2.4c). By applying the inverse Fourier transform, with the use of the convolution theorem, this allows us to mathematically reduce the problem to the system of integral equations with respect to unknown functions $g_x(x)$ and $g_y(x)$, which is valid over the crack length:

$$\begin{cases} \int_a^b K_{11}(x-\xi)g_x(\xi)d\xi+\int_a^b K_{12}(x-\xi)g_y(\xi)d\xi=f_1(x) \\ \int_a^b K_{21}(x-\xi)g_x(\xi)d\xi+\int_a^b K_{22}(x-\xi)g_y(\xi)d\xi=f_2(x) \end{cases},\qquad a\le x\le b \tag{2.13}$$

where

$$K_{11}(x)=2\int_0^\infty\left\{\left[s^3h^2-s^2h+\frac{s}{2}\right]e^{-2sh}-\frac{s}{2}\right\}\cos(xs)\,ds=\frac{1}{x^2}+$$
$$+\frac{12h^2(16h^4-24h^2x^2+x^4)}{(16h^4+8h^2x^2+x^4)(4h^2+x^2)^2}-8h^2\frac{4h^2-3x^2}{(4h^2+x^2)^3}+\frac{4h^2-x^2}{(4h^2+x^2)^2} \tag{2.14a}$$

$$K_{12}(x)=-2h^2\int_0^\infty s^3e^{-2sh}\sin(xs)\,ds=-\frac{96h^3x(4h^2-x^2)}{(16h^4+8h^2x^2+x^4)(4h^2+x^2)^2}, \tag{2.14b}$$
$$K_{21}(x)=-K_{12}(x)$$

$$K_{22}(x)=2\int_0^\infty\left\{\left[s^3h^2+s^2h+\frac{s}{2}\right]e^{-2sh}-\frac{s}{2}\right\}\cos(xs)\,ds=\frac{1}{x^2}+$$
$$+\frac{12h^2(16h^4-24h^2x^2+x^4)}{(16h^4+8h^2x^2+x^4)(4h^2+x^2)^2}+8h^2\frac{4h^2-3x^2}{(4h^2+x^2)^3}+\frac{4h^2-x^2}{(4h^2+x^2)^2} \tag{2.14c}$$

$$f_1(x)=\frac{hP_0}{\mu(1-c^2)}\int_0^\infty se^{-sh}\sin[(x-x_0)s]\,ds=\frac{2h^2(x-x_0)P_0}{\mu(1-c^2)[h^2+(x-x_0)^2]^2} \tag{2.15a}$$

$$f_2(x)=-\frac{P_0}{\mu(1-c^2)}\int_0^\infty(sh+1)e^{-sh}\cos[(x-x_0)s]\,ds=$$
$$=-\frac{2h^3P_0}{\mu(1-c^2)[h^2+(x-x_0)^2]^2} \tag{2.15b}$$

3. General Properties of the Direct and the Inverse Problems

The direct problem can be formulated as follows. If we know completely the geometry of the crack, i.e. quantities h,a,b, and the applied force, i.e. quantities P_0 and x_0, then we can solve the system of integral equations (2.13) and determine all physical characteristics of the problem. The most physically important one is the settlement of the upper boundary surface, i.e. function $u_y^+(x,h)$. This can be directly extracted from Eq.(2.9) which in terms of functions $g_x(x)$ and $g_y(x)$ is

$$f_0(x)=u_y^+(x,h)-u_y^{0+}(x,h)=\frac{h}{\pi}\int_a^b g_x(\xi)\,d\xi\int_0^\infty se^{-sh}\sin[(x-\xi)s]ds+$$
$$+\frac{1}{\pi}\int_a^b g_y(\xi)\,d\xi\int_0^\infty(1+sh)e^{-sh}\cos[(x-\xi)s]ds= \tag{3.1}$$
$$=\frac{2h^2}{\pi}\int_a^b\frac{g_x(\xi)(x-\xi)\,d\xi}{[h^2+(x-\xi)^2]^2}+\frac{2h^3}{\pi}\int_a^b\frac{g_y(\xi)\,d\xi}{[h^2+(x-\xi)^2]^2}$$

where we have written the final result for the function, which is the difference between the settlement in the problem under consideration and the one in the problem free of crack. Therefore, the represented function (3.1) corresponds to contribution of the crack to the value of the settlement.

Now, some words about qualitative properties of system (2.13). It can easily be seen that the diagonal kernels, functions $K_{11}(x)$ and $K_{22}(x)$ have hyper-singular behavior at the origin. A stable method to solve hyper-singular integral equations was proposed in the work [11]. Let us rewrite system (2.13) as

$$\begin{cases} \int\limits_a^b \left[\dfrac{1}{(x-\xi)^2} + K_{11}^0(x-\xi) \right] g_x(\xi) d\xi + \int\limits_a^b K_{12}(x-\xi) g_y(\xi) d\xi = f_1(x) \\ \int\limits_a^b K_{21}(x-\xi) g_x(\xi) d\xi + \int\limits_a^b \left[\dfrac{1}{(x-\xi)^2} + K_{22}^0(x-\xi) \right] g_y(\xi) d\xi = f_2(x) \end{cases}, \quad a \le x \le b \qquad (3.2)$$

in the form where we have explicitly extracted the characteristic hyper-singular part of the diagonal kernels, and the superscripts designate some regular functions. Then briefly speaking our numerical algorithm can be described as follows.

First of all, we subdivide full interval (a,b) to a set of n small elementary subintervals of the same length $\varepsilon = (b-a)/n$, by nodes $a = \xi_0, \xi_1, \xi_2, ..., \xi_{n-1}, \xi_n = b,\ \xi_j = a + j\varepsilon, (j = 0,1,...,n)$. The central point of each sub-interval (ξ_{i-1}, ξ_i) is designated as x_i, hence $x_i = a + (i - 1/2)\varepsilon, (i = 1,...,n)$. It can be proved (see [11]) that a correct approximation to a bounded solution of system (3.2) can be constructed as a solution to the following linear algebraic system $(i = 1,...,n)$:

$$\begin{cases} \sum\limits_{j=1}^{n} \left[\dfrac{1}{x_i - \xi_j} - \dfrac{1}{x_i - \xi_{j-1}} + \varepsilon K_{11}^0(x_i - \xi_j) \right] g_x(\xi_j) + \varepsilon \sum\limits_{j=1}^{n} K_{12}(x_i - \xi_j) g_y(\xi_j) = f_1(x_i) \\ \varepsilon \sum\limits_{j=1}^{n} K_{21}(x_i - \xi_j) g_x(\xi_j) + \sum\limits_{j=1}^{n} \left[\dfrac{1}{x_i - \xi_j} - \dfrac{1}{x_i - \xi_{j-1}} + \varepsilon K_{22}^0(x_i - \xi_j) \right] g_y(\xi_j) = f_2(x_i) \end{cases} \qquad (3.3)$$

The inverse problem can be formulated as follows. Let a known outer force $P_0(x)\delta(x - x_0)$ be applied to the boundary surface of the elastic half-space (2D problem). This implies that quantities $P_0(x)$ and x_0 are known. Let us assume that there is a horizontal crack in the half-space (see Fig.1), but its position and geometry are unknown. Let the shape of the boundary surface, function $f_0(x)$ in Eq.(3.1), be known over some finite-length set Γ of the boundary line $y = h$ as a certain input data, which is obtained for example, from the results of the experimental measurement. Then our goal is to reconstruct crack's geometry. In frames of such an approach, in Eqs. (2.13) – (2.15) and (3.1) quantities $P_0, x_0, \mu, c^2, \Gamma, f_0(x), (x \in \Gamma)$ are known, and quantities $h, a, b, g_x(x), g_y(x), (a \le x \le b)$ are unknown.

Speaking about a stable algorithm, in order to solve numerically the posed inverse problem, let us note that the input data, function $f_0(x)$, may be given only approximately, with a certain error. Hence, the algorithm to solve this inverse problem should be stable with respect to small perturbations of function $f_0(x)$.

The algorithm, which is presented in this work, is based on the discrete analogue (3.3) of the basic system of hyper-singular integral equations (3.2). Let us write linear algebraic system (3.3) in the matrix form:

$$Ag = f, \quad A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad A_{11} = \left(a_{11}^{ij}\right), \quad A_{12} = \left(a_{12}^{ij}\right), \quad A_{21} = \left(a_{21}^{ij}\right), \quad A_{22} = \left(a_{22}^{ij}\right),$$

$$g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}, \; g_1 = \left(g_1^j\right), \; g_2 = \left(g_2^j\right); \quad f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \; f_1 = \left(f_1^i\right), \; f_2 = \left(f_2^i\right), \quad i, j = 1, \ldots, n. \tag{3.4}$$

where the matrices and the right-hand sides are

$$a_{11}^{ij} = \frac{1}{x_i - \xi_j} - \frac{1}{x_i - \xi_{j-1}} + \varepsilon K_{11}^0 (x_i - \xi_j), \quad a_{12}^{ij} = \varepsilon K_{12}(x_i - \xi_j), \quad a_{21}^{ij} = -a_{12}^{ij},$$

$$a_{22}^{ij} = \frac{1}{x_i - \xi_j} - \frac{1}{x_i - \xi_{j-1}} + \varepsilon K_{22}^0 (x_i - \xi_j), \quad f_1^i = f_1(x_i), \quad f_2^i = f_2(x_i) \tag{3.5}$$

Obviously, matrix A and right-hand side f both depend on parameters h, a, b: $A = A(h, a, b)$, $f = f(h, a, b)$.

If function $f_0(x)$ is known as the input data on the set Γ then one can choose a finite discrete array of nodes on this set: $x_k \in \Gamma, (k = 1, \ldots, m)$, to treat relation (3.1) expressing results of the measurements on a discrete grid. By replacing the integral there by an integral sum, one can approximate this relation by the system of algebraic equalities, which can be written in the symbolic form as follows

$$Bg = d, \; B = \begin{pmatrix} B_1 & B_2 \end{pmatrix}, \; B_1 = \left(b_1^{kj}\right), \; B_2 = \left(b_2^{kj}\right), \; d = \left(d_k\right), \; (k = 1, \ldots, m; \; j = 1, \ldots, n)$$

$$b_1^{kj} = \frac{2h^2 \varepsilon (x_k - \xi_j)}{\pi [h^2 + (x_k - \xi_j)^2]^2}, \quad b_2^{kj} = \frac{2h^3 \varepsilon}{\pi [h^2 + (x_k - \xi_j)^2]^2}, \quad d_k = f_0(x_k) \tag{3.6}$$

The inversion of matrix equation (3.4): $g = (A^{-1} f)(h, a, b)$, where we explicitly indicate that the inversion operator depends on geometrical parameters h, a, b, and the substitution of the obtained relation to Eq.(3.6) reduces this inverse problem in its discrete form to the algebraic relation

$$(BA^{-1} f)(h, a, b) = d \tag{3.7}$$

It should be noted that dimension of matrix A is $2n \times 2n$, hence dimension of matrix BA^{-1} is $m \times 2n$. Therefore, Eq.(3.7) is written correctly since both its sides are vectors of dimension m. Our approach to construct a stable solution to operator equation (3.7) is founded on the minimization of the discrepancy functional

$$\min \Omega(h, a, b), \quad \Omega = \left\| (BA^{-1} f)(h, a, b) - d \right\|_{l_2}^2 = \sum_{k=1}^{m} \left[(BA^{-1} f)_k (h, a, b) - d_k \right]^2 \tag{3.8}$$

which is again a functional of three parameters h, a, b.

4. Numerical Treatment and Examples of the Reconstruction

The minimization of functional (3.8) can be attained by any classical method of optimization (see, for example, [12]). The main restriction of regular iterative schemes is that they give only a local minimum of respective functionals. Another difficulty is connected with a non-uniqueness of the solution. It is not evident that a local minimum is its global minimum, which in the case of exact input data is zero.

We tested application of the algorithm proposed in [13] to our inverse problem. We aimed at a search of global minimum by a global random search. This algorithm is developed to seek maxima, but it can be applied to minima too. Efficiency of the algorithm is explained by the two following specific features: 1) random sampling of values in the neighborhood of the points, for which the values of the functional are small, happens more frequently than that in the neighborhood of the points, where the values of the functional are large, and 2) domains, in which random values of variables are chosen, are gradually contracted to small neighborhoods of the points with small values of the functional. This technique demonstrates remarkable convergence for all considered examples.

For all examples the input data is taken from the solution of respective direct problem. It should be noted that, in order to generate approximate (i.e. inexact) input data, we numerically solved the direct problem, and then perturbed the so obtained results by random quantities, in accordance with the assigned "error" of the input data. For all examples demonstrated below we used $m=100$ points of measurements, to form the array of the input data, so that $x_k = -5 + 0.1(k-1/2), k = 1,...,m$, $x_k \in (-5,5)$. It is clear that with such a choice of the trial points they form a uniform set around the applied force $P_0(x)\delta(x)$, $(x_0 = 0)$.

We have performed numerous calculations and a thorough numerical investigation for many examples. Some results on crack's identification are presented in Table 1. For all examples below $\ell = b - a$ and $c^2 = \mu/(\lambda + 2\mu) = 0.3$.

Table 1

input data error	h	a	ℓ	type of result
	1.000	-0.500	1.000	exact
0%	1.005	-0.490	0.997	restored
	2.000	1.500	1.000	exact
0%	1.997	1.504	0.997	restored
	0.500	3.500	1.500	exact
0%	0.490	3.495	1.509	restored
	3.000	4.000	0.500	exact
0%	2.988	4.003	0.500	restored

The physical conclusions from Table 1 are quite evident. Then we studied the stability of the proposed algorithm if the input data is given with a certain error.

Table 2

input data error	h	a	ℓ	type of result
	1.500	-0.500	1.000	exact
5%	1.496	-0.500	1.000	restored
	2.500	1.000	1.500	exact
5%	2.502	0.982	1.495	restored
	4.000	-3.000	4.500	exact
5%	4.014	-3.009	4.501	restored
	0.200	2.000	0.100	exact
5%	0.208	2.003	0.096	restored

Some examples with more significant error in the input data are presented below.

Table 3

input data error	h	a	ℓ	type of result
	8.000	-0.500	1.000	exact
15%	8.031	-0.532	1.001	restored
	6.000	2.500	4.000	exact
15%	6.220	2.620	4.116	restored
	4.000	-3.500	8.000	exact
15%	4.032	-3.521	7.996	restored
	2.000	4.500	0.200	exact
15%	1.938	4.497	0.198	restored

Table 4

input data error	h	a	ℓ	type of result
	1.000	-1.000	2.000	exact
25%	0.927	-1.013	2.037	restored
	7.000	-4.500	9.000	exact
25%	7.129	-4.596	9.076	restored
	0.300	1.500	0.100	exact
25%	0.316	1.492	0.103	restored
	1.500	3.000	0.500	exact
25%	1.516	3.038	0.474	restored

It is very interesting to analyze the influence of the input data error for a chosen single crack. In Table 5 below we demonstrate such dependence for the last crack taken from Table 4.

Table 5

input data error	h	a	ℓ	type of result
	1.500	3.000	0.500	exact
0%	1.515	3.004	0.498	restored
5%	1.513	2.981	0.497	restored
15%	1.490	2.983	0.514	restored
25%	1.516	3.038	0.474	Restored

The following evident conclusions follow from the presented numerical results:

1. The proposed algorithm demonstrates a wonderful efficiency. Even with very rough precision of the input data the reconstruction is very good. Moreover, for approximate input data precision of the reconstruction is higher than that of the input data. This clearly confirms the stability of the algorithm.

2. Concerning the comparison between various examples presented, taking into account that all of them are of high precision, we can however analyze the cases when the reconstruction is highly precise or less precise. Regardless precision of the input data, long cracks are reconstructed more accurately than small ones.

3. We have also tried to vary the length of the intervals where the input data is collected. Under such numerical experiments we could conclude that precision of the reconstruction improves when the measure of the set Γ increases, that is quite natural from the physical point of view.

4. A numerous number of numerical experiments performed by the authors indicate that in the cases when the object under reconstruction is situated closer to the boundary interval Γ, the precision of the reconstruction is higher.

5. The present consideration is restricted by cracks parallel to the free boundary surface of the half-space only. The convenience of the present treatment consists of the fact that this is based on the classical Fourier transform. A different approach may be applied for arbitrary crack orientation in the half-space; this will be the subject of the authors' next work.

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