ՀԱՅԱՍՏԱՆԻ ԳԻՏՈՒԹՅՈՒՆՆԵՐԻ ԱԶԳԱՅԻՆ ԱԿԱԴԵՄԻԱՅԻ ՏԵՂԵԿԱԳԻՐ
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ASYMPTOTIC THEORY OF ANISOTROPIC PLATES AND SHELLS

Lenser Aghalovyan*)

Key words: asymptotic method, elasticity, anisotropic plate, beam, shell, vibrations, space problem.
Ключевые слова: асимптотический метод, упругость, анизотропия, балка, пластина, оболочка, колебания, трехмерная задача.

Լ.Ա. Աղալովյան
Անիզոտրոպ սալերի և թաղանթների ասիմպտոտիկ տեսություն

Աշխատանքն ունի ակնարկային բնույթ ուր շարադրված են սինգուլյար գրգռված դիֆերենցիալ հավասարումների լուծման ասիմպտոտիկ մեթոդը և այդ մեթոդի կիրառությունը բարակ մարմինների (հեծաններ, սալեր, թաղանթներ) դասական և ոչ դասական եզրային խնդիրները լուծելու համար: Ստացված ընդհանուր ասիմպտոտիկ լուծումներն ուղեկցվում են որոշ դասի խնդիրների ճշգրիտ լուծումներով:

Л.А. Агаловян
Асимптотическая теория анизотропных пластин и оболочек

Работа носит обзорный характер, где изложены суть асимптотического метода решения сингулярно возмущенных дифференциальных уравнений и методика применения этого метода для решения статических и динамических краевых задач тонких тел (балки, пластины, оболочки). Рассмотрены как классические, так и неклассические краевые задачи. Более общие результаты проиллюстрированы решениями конкретных задач.

The essence of asymptotic method solution of singularly-perturbed differential equations is explained. The mentioned method is applying for the boundary-value problems of statics and dynamics of thin bodies (beams, plates, shells) solving. The general results is illustrated by the solutions of determined classes problems.

I. *The solution of the first static boundary value problem of thermoelasticity for beams, bars, plates and shells.*

Before describing the essence of the asymptotic method of plane and space problems solutions of elasticity theory for beams, plates and shells, we shall find out what kind of perturbed by small (big) parameters differential equations correspond to these thin bodies.

1. Regularly and singularly perturbed differential equations and the asymptotic method of their solution. All the differential equations, containing a small parameter, are divided into regularly perturbed and singularly perturbed equations. In order to reveal their principal difference and the application singularities of the asymptotic method for their solution, we consider the following two model equations:

a) $$u''+\varepsilon u'=0,\qquad u=u(x),\qquad x\in[0,1] \tag{1.1}$$

b) $$\varepsilon u''+u'=0 \tag{1.2}$$

where ε is a small parameter. It is required to find the solution of equations (1.1), (1.2) under the boundary conditions

$$u(0)=\alpha,\qquad u(1)=\beta\,. \tag{1.3}$$

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$$u = C_1 + C_2 e^{-x\varepsilon} \tag{1.4}$$

which is the continuous function from small parameter ε, will be the solution of equation (1.1). Satisfying conditions (1.3) we have the solution

$$\begin{aligned} u &= \alpha + (\beta - \alpha)x \qquad \text{at } \varepsilon = 0 \\ u &= \frac{1}{e^{\varepsilon(1+x)} - e^{\varepsilon x}}\left[\beta\left(e^{\varepsilon(1+x)} - e^{\varepsilon}\right) - \alpha\left(e^{\varepsilon x} - e^{\varepsilon}\right)\right] \qquad \text{at } \varepsilon \neq 0 \end{aligned} \tag{1.5}$$

it's easy to verify the continuousness of the solution at $\varepsilon = 0$.

As equation (1.1) contains a small parameter, it is natural to use the asymptotic method and sick the solution in the form of a power series

$$u = \varepsilon^s u_s, \qquad s = \overline{0,\infty} \tag{1.6}$$

where signification $s = \overline{0,\infty}$ means summing by umbral (repeating) index s from zero to $(+\infty)$. Substituting (1.6) into (1.1) we get an iteration equation

$$u_s'' + u_{s-1}' = 0, \qquad u_m \equiv 0 \text{ at } m < 0 \tag{1.7}$$

for determining coefficients u_s.

Conditions (1.3) will have the form

$$u_0(0) = \alpha, \quad u_0(1) = \beta, \qquad u_s(0) = 0, \; u_s(1) = 0, \; s \geq 1 \tag{1.8}$$

At $s = 0$ equation (1.7) will have the form

$$u_0'' = 0 \tag{1.9}$$

i.e. in case of regularly perturbed equation (small parameter ε is not the coefficient of the big derivative), the shortened (not perturbed) equation, i.e. equation (1.1) at $\varepsilon = 0$, has the same order, which the initial equation (1.1) has. This important property permits us to satisfy the given boundary conditions. Particularly, at $s = 0$ we have

$$u_0 = (\beta - \alpha)x + \alpha \tag{1.10}$$

and at $s = 1, 2$, taking into account conditions (1.8), the solutions

$$u_1 = \frac{1}{2}(\alpha - \beta)x(x-1), \qquad u_2 = -\frac{1}{12}(\alpha - \beta)x(x-1)(2x-1) \tag{1.11}$$

The iteration process may be continued and got the solution for any approach. Later on, the question of series (1.6) similarity is considered. As a rule, the similarity is asymptotical, i.e. the error is of the first rejected term order of the series.

The property, illustrated on the boundary value problem (1.1), (1.3), is common for all the regularly perturbed equations including for the equations in private derivatives, therefore such equations may be solved using the decomposition of type (1.6).

Now we consider singularly perturbed equation (1.2), i.e. when the small parameter is the coefficient of he highest operator (derivative). The solution of equation (1.2) is

$$u = A_1 + A_2 e^{-x/\varepsilon} \tag{1.12}$$

which is not the continuous function from ε already. Satisfying conditions (1.3) we get

$$u = \frac{1}{1-e^{-1/\varepsilon}}\left[\beta - \alpha e^{-1/\varepsilon} + (\alpha-\beta)e^{-x/\varepsilon}\right] \tag{1.13}$$

At $\varepsilon << 1$ $u \approx \beta$ out of dependence on values x, except some small area near $x = 0$, which is called boundary layer. The corresponding graphs are depicted in fig.1, fig.2.

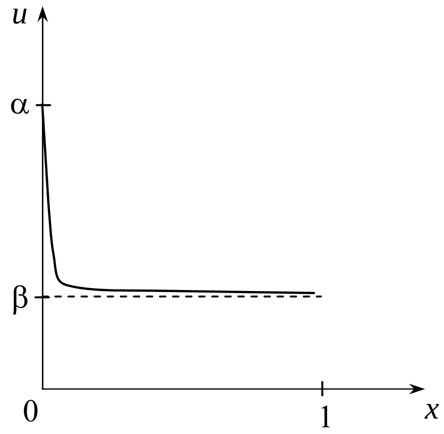

Fig. 1

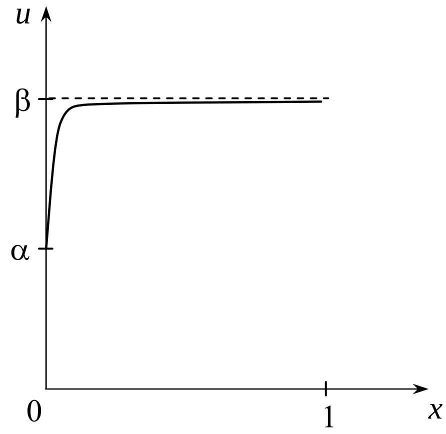

Fig. 2

Consider a possibility of the boundary value problem solution (1.2), (1.3) by an asymptotic method. Naturally, the solution, in this case too, is sought in the form of (1.6). For u_s we obtain the equation

$$u''_{s-1} + u'_s = 0 \tag{1.14}$$

At $s = 0$ we have

$$u'_0 = 0 \qquad \text{or} \qquad u_0 = C_1 = \text{const} \tag{1.15}$$

i.e. the unperturbed equation has less order, than the perturbed (1.2). That's why by solution (1.15) it is not possible to satisfy two conditions (1.3). A question rises – which of these conditions should be satisfied. From the above brought analysis of the exact solution, the satisfaction of the second of the conditions (1.3) becomes natural, i.e. the condition, near the end of which there is no boundary layer. Then $C_1 = \beta$ and $u_0 = \beta$. It appears that it is possible to satisfy the first condition of (1.3) too, if the solution of the boundary layer for the end $x = 0$ is built. For it replacement of variable $t = -x/\varepsilon$ is introduced and such solution of transformed equation (1.2) which has fading nature and can remove arising residual at $x = 0$ [1-4]

$$u_b = C_2 e^t = C_2 e^{-x/\varepsilon} \tag{1.16}$$

will be this solution. Requiring at $x = 0$ $(t = 0)$

$$u_0(x=0) + u_b(t=0) = \alpha \tag{1.17}$$

we determine $C_2 = \alpha - \beta$. As a result the initial approach will correspond to solution

$$u^{(0)} = u_0 + u_b = \beta + (\alpha - \beta)e^{-x/\varepsilon} \tag{1.18}$$

which at small ε practically coincides with the exact solution (1.13). If at determination u_0 we satisfy the first condition of (1.3), we get $u_0 = \alpha$. Then it is impossible to satisfy the condition at $x = 1$, as there is no boundary layer there, because introducing the replacement of the variable $\eta = (x-1)/\varepsilon$, $-1/\varepsilon \le \eta \le 0$, equation (1.2) is transformed into $u''_\eta + u'_\eta = 0$, which does not have fading solution on the interval $-1/\varepsilon \le \eta \le 0$. The iteration process may be continued. In this way it is possible to give mathematical proof of the procedure of finding asymptotic solution.

From the above built asymptotic solution the conclusions general for singularly perturbed differential equations follow: the solution cannot be obtained on the form of one decomposition by small parameter of (1.6) type, it is made up from the principal solution (I^{int}) and the solutions for the boundary layers (I_b); several boundary layers may exist in dependence of the problem and order of the perturbed operator; these solutions may be built separately and product their conjugation with the help of the boundary conditions.

In the problems of elasticity theory for thin bodies in the equations the small parameter is the coefficient of not the whole highest operator, but of its part, yet the structure of the solution remains unchangeable $(I = I^{int} + I_b)$. The unperturbed equation has the smallest space dimension and the boundary functions constitute a countable set.

2. The asymptotics of problems solutions of bend of beams and tension-compression of bars. Classical theory of beams and bars is built on the base of Bernoulli-Coulomb-Euler plane cross-sections hypothesis. Kirchhoff generalized this hypothesis (hypothesis of undeformable normals) for derivation of two-dimensional equations of plates and by the variation method developed the well-known boundary condition for the free end. Love applied the hypothesis of undeformable normals for deducing the equations of shell. The classical theory of shells obtained a complete form thanks to S.P.Timoshenko, V.Flugge, V.Z.Vlasov, A.L.Goldenweiser, A.I.Lurier, V.V.Novoghilov monographs. The classical theory of anisotropic plates, including layered plates, is built by S.G.Lekhnitski, and the theory for anisotropic shells was built by S.A.Ambartsumyan. And with this sequence we discuss the problem of reduction of the corresponding three-dimensional problems of elasticity theory to two-dimensional and one-dimensional problems of mathematical physics, reveal the connection of such reduction with classical theory of beams, plates and shells. By asymptotic method we solve new classes of problems for thin bodies, which are not possible to solve on the base of the classical theory hypothesis.

We set the problem: to find the solution of the first static boundary value problem of thermoelasticity in rectangular domain $D = \{(x,y): 0 \le x \le l,\ -h \le y \le h,\ h << l\}$ with the account of volume forces and temperature by Duhamel-Neumann model. It is necessary to find the solution:

of the equilibrium equations

$$\begin{aligned} &\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + F_x(x,y) = 0 \\ &\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + F_y(x,y) = 0 \end{aligned} \tag{2.1}$$

of the equations of state (Hook's generalized law)

$$\frac{\partial u}{\partial x}=\frac{1}{E}(\sigma_{xx}-\nu\sigma_{yy})+\alpha_{11}\theta$$
$$\frac{\partial \mathrm{v}}{\partial y}=\frac{1}{E}(\sigma_{yy}-\nu\sigma_{xx})+\alpha_{22}\theta \qquad (2.2)$$
$$\frac{\partial u}{\partial y}+\frac{\partial \mathrm{v}}{\partial x}=\frac{1}{G}\sigma_{xy}+\alpha_{12}\theta$$

where α_{ik} are the coefficients of thermal conductivity, E, G are Young's and shear modules, ν is Poisson's ratio, $\theta=T(x,y)-T_0(x,y)$ is the change of temperature under boundary conditions on the longitudinal ends $y=\pm h$

$$\sigma_{xy}(x,\pm h)=\pm X^{\pm}(x), \qquad \sigma_{yy}(x,\pm h)=\pm Y^{\pm}(x) \qquad (2.3)$$

and under the conditions at $x=0,l$ (conditions of fastening), which are considered to be arbitrary for the present. As mass forces, for example, weight ($F_x=0$, $F_y=-\rho(x,y)g$) or the reduced seismic force by Mononobe model ($F_x=\beta k_s P$, $F_y\approx 0.75F_x$) may come forward, where ρ is the density, β is the coefficient of dynamics, k_s is the coefficient of seismicity, P is the weight of the rectangle.

For solving the set problem we pass to dimensionless coordinates and displacements

$$x=l\xi, \quad y=h\zeta, \quad U=u/l, \; V=\mathrm{v}/l \qquad (2.4)$$

Equations (2.1), (2.2) will have the form

$$\frac{\partial\sigma_{xx}}{\partial\xi}+\varepsilon^{-1}\frac{\partial\sigma_{xy}}{\partial\zeta}+lF_x(l\xi,h\zeta)=0, \qquad \varepsilon=h/l$$
$$\frac{\partial\sigma_{xy}}{\partial\xi}+\varepsilon^{-1}\frac{\partial\sigma_{yy}}{\partial\zeta}+lF_y(l\xi,h\zeta)=0$$
$$\frac{\partial U}{\partial\xi}=\frac{1}{E}(\sigma_{xx}-\nu\sigma_{yy})+\alpha_{11}\theta \qquad (2.5)$$
$$\varepsilon^{-1}\frac{\partial V}{\partial\zeta}=\frac{1}{E}(\sigma_{yy}-\nu\sigma_{xx})+\alpha_{22}\theta$$
$$\varepsilon^{-1}\frac{\partial U}{\partial\zeta}+\frac{\partial V}{\partial\xi}=\frac{1}{G}\sigma_{xy}+\alpha_{12}\theta$$

System (2.5) is singularly perturbed by small parameter ε , but this singularity differs from the classical singularity (1.2), as the small (big) parameter is not the coefficient of the whole highest operator (derivative), but only part of it. The structure of the solution, as we shall be convinced of it below, remains unchangeable

$$I=I^{int}+I_b \qquad (2.6)$$

i.e. the solution consists of the solutions of the inner (basic) problem and the problem for the boundary layer. In our case $I_b=I_b^{(1)}+I_b^{(2)}$, where $I_b^{(1)}$ is the solution of the boundary layer for the end $x=0$, and $I_b^{(2)}$ is for the end $x=l$. Note that such a structure of solution (2.6) takes place in any physical problem, considered in narrow area.

It is known that not all the components of the stresses tensor and displacement vector have the same contribution in the common stress-strain state. By virtue of it when determining the solution, particularly, of the inner problem, the question of the correct determination of asymptotic order of sought values is really very important and considerably difficult. It has deeper roots, as having formulated any physical law or hypothesis, some asymptotics is given in fact. The correct determination of the asymptotics is considered to be art by some authors [6].

The solution of the inner problem I^{int} will be sought in the form of

$$I^{int} = \varepsilon^{q_I+s} I^{(s)}(\xi,\zeta) \qquad s=\overline{0,N} \tag{2.7}$$

where I is any of the stresses and displacements, $s=\overline{0,N}$ means summing by repeated (dummy) index s from the zero up to the number of approaches N. It is established [5,7], that after the substitution of (2.7) into the system (2.5) and equalizing the coefficients at the same degrees of ε in each equation, we get incontradictory system for determining $I^{(s)}$, if

$$q_{\sigma_{xx},u} = -2, \quad q_{\sigma_{xy}} = -1, \quad q_{\sigma_{yy}} = 0, \quad q_{\mathrm{v}} = -3 \tag{2.8}$$

Having solved with the account of (2.7), (2.8) obtained from (2.5) a system we have

$$V^{(s)} = \mathrm{v}^{(s)}(\xi) + \mathrm{v}_*^{(s)}(\xi,\zeta)$$

$$U^{(s)} = -\frac{d\mathrm{v}^{(s)}}{d\xi}\zeta + u^{(s)}(\xi) + u_*^{(s)}(\xi,\zeta)$$

$$\sigma_{xx}^{(s)} = -E\frac{d^2\mathrm{v}^{(s)}}{d\xi^2}\zeta + E\frac{du^{(s)}}{d\xi} + \sigma_{x*}^{(s)} \tag{2.9}$$

$$\sigma_{xy}^{(s)} = \frac{1}{2}E\frac{d^3\mathrm{v}^{(s)}}{d\xi^3}\zeta^2 - E\frac{d^2u^{(s)}}{d\xi^2}\zeta + \sigma_{xy0}^{(s)}(\xi) + \sigma_{xy*}^{(s)}(\xi,\zeta)$$

$$\sigma_{yy}^{(s)} = -\frac{1}{6}E\frac{d^4\mathrm{v}^{(s)}}{d\xi^4}\zeta^3 + \frac{1}{2}E\frac{d^3u^{(s)}}{d\xi^3}\zeta^2 - \frac{d\sigma_{xy0}^{(s)}}{d\xi} + \sigma_{yy0}^{(s)}(\xi) + \sigma_{y*}^{(s)}(\xi,\zeta)$$

where

$$\mathrm{v}_*^{(s)} = \int_0^\zeta \left[\frac{1}{E}\left(\sigma_{yy}^{(s-4)} - \nu\sigma_{xx}^{(s-2)}\right) + \alpha_{22}\theta^{(s-2)}\right]d\zeta$$

$$u_*^{(s)} = \int_0^\zeta \left[\frac{1}{G}\sigma_{xy}^{(s-2)} + \alpha_{12}\theta^{(s-1)} - \frac{\partial \mathrm{v}_*^{(s)}}{\partial\xi}\right]d\zeta$$

$$\sigma_{x*}^{(s)} = E\frac{\partial u_*^{(s)}}{\partial\xi} + \nu\sigma_{yy}^{(s-2)} - E\alpha_{11}\theta^{(s)} \tag{2.10}$$

$$\sigma_{xy*}^{(s)} = -\int_0^\zeta \left(F_x^{(s)} + \frac{\partial\sigma_{x*}^{(s)}}{\partial\xi}\right)d\zeta, \qquad \sigma_{y*}^{(s)} = -\int_0^\zeta \left(F_y^{(s)} + \frac{\partial\sigma_{xy*}^{(s)}}{\partial\xi}\right)d\zeta$$

$$F_x^{(0)} = l\varepsilon^2 F_x(l\xi, h\zeta), \quad F_y^{(0)} = l\varepsilon F_y(l\xi, h\zeta), \qquad F_x^{(s)} = F_y^{(s)} = 0, \ s \neq 0$$

$$\theta^{(0)} = \varepsilon^2\theta, \quad \theta^{(s)} = 0, \ s \neq 0, \qquad I^{(m)} \equiv 0 \ \text{at} \ m < 0$$

Solution (2.9) contains unknown functions $u^{(s)}$, $\text{v}^{(s)}$, $\sigma_{xy0}^{(s)}$, $\sigma_{yy0}^{(s)}$, which must be determined from conditions (2.3) and the conditions at $x = 0, l$. Having satisfied (2.3) $\sigma_{xy0}^{(s)}$, $\sigma_{yy0}^{(s)}$ are expressed through $u^{(s)}$, $\text{v}^{(s)}$

$$\sigma_{xy0}^{(s)} = -\frac{1}{2}E\frac{d^3\text{v}^{(s)}}{d\xi^3} + \frac{1}{2}\left(X^{+(s)} - X^{-(s)}\right) - \frac{1}{2}\left(\sigma_{xy*}^{(s)}(\xi,1) + \sigma_{xy*}^{(s)}(\xi,-1)\right)$$

$$\sigma_{yy0}^{(s)} = \frac{1}{2}\left(Y^{+(s)} - Y^{-(s)}\right) - \frac{1}{2}\frac{dq_x^{(s)}}{d\xi} - \frac{1}{2}\left(\sigma_{y*}^{(s)}(\xi,1) + \sigma_{y*}^{(s)}(\xi,-1)\right) \tag{2.11}$$

$$X^{\pm(0)} = \varepsilon X^{\pm}, \quad Y^{\pm(0)} = Y^{\pm}, \qquad X^{\pm(s)} = Y^{\pm(s)} = 0, \ s \neq 0$$

For determining functions $u^{(s)}$, $\text{v}^{(s)}$

$$E\frac{d^2u^{(s)}}{d\xi^2} = q_x^{(s)}$$

$$q_x^{(s)} = -\frac{1}{2}\left(X^{+(s)} + X^{-(s)}\right) + \frac{1}{2}\left(\sigma_{xy*}^{(s)}(\xi,1) - \sigma_{xy*}^{(s)}(\xi,-1)\right)$$

$$\frac{1}{3}E\frac{d^4\text{v}^{(s)}}{d\xi^4} = q^{(s)} \tag{2.12}$$

$$q^{(s)} = \frac{1}{2}\left(Y^{+(s)} + Y^{-(s)}\right) - \frac{1}{2}\left(\sigma_{y*}^{(s)}(\xi,1) - \sigma_{y*}^{(s)}(\xi,-1)\right) +$$

$$+\frac{1}{2}\frac{d}{d\xi}\left[X^{+(s)} - X^{-(s)} - \sigma_{xy*}^{(s)}(\xi,1) - \sigma_{xy*}^{(s)}(\xi,-1)\right]$$

equations are obtained.

From (2.12) we have

$$Eu^{(s)} = \int_0^{\xi} d\xi \int_0^{\xi} q_x^{(s)} d\xi + C_1^{(s)}\xi + C_2^{(s)}$$

$$\frac{1}{3}E\text{v}^{(s)} = \int_0^{\xi} d\xi \int_0^{\xi} d\xi \int_0^{\xi} d\xi \int_0^{\xi} q^{(s)} d\xi + C_3^{(s)}\frac{\xi^3}{3!} + C_4^{(s)}\frac{\xi^2}{2} + C_5^{(s)}\xi + C_6^{(s)} \tag{2.13}$$

Comparing solution (2.13) with (2.9), the stresses will contain constants $C_1^{(s)}$, $C_3^{(s)}$, $C_4^{(s)}$, constants $C_2^{(s)}$, $C_5^{(s)}$, $C_6^{(s)}$ will characterize rigid displacements, which may be excluded fastening one point, for example the origin of coordinates, requiring

$$U^{(s)}(0,0) = 0, \ V^{(s)}(0,0) = 0, \ \left(\frac{\partial U^{(s)}}{\partial \zeta} - \frac{\partial V^{(s)}}{\partial \xi}\right)_{\substack{\xi=0 \\ \zeta=0}} = 0 \tag{2.14}$$

Only by solution (2.7)-(2.13) it is impossible to satisfy the conditions at $x = 0, l$, for example, conditions like

$$\begin{aligned} &\text{a) } \sigma_{xx}(0,\zeta) = \varphi(\zeta), \quad \sigma_{xy}(0,\zeta) = \psi(\zeta) \\ &\text{b) } u(0,\zeta) = \varphi_1(\zeta), \quad \text{v}(0,\zeta) = \psi_1(\zeta) \end{aligned} \tag{2.15}$$

In better case we can satisfy conditions (2.15) in one or two points, but not in all the points with unknown yet constants $C_1^{(s)}$, $C_3^{(s)}$, $C_4^{(s)}$ of the solution, this again proves the singular perturbance of the initial boundary problem. In order to satisfy conditions (2.15), it is necessary to built the solution of the boundary layer at $x = 0$.

As inhomogeneous equations (2.5) and boundary conditions (2.3) are satisfied by the solution of the inner problem, the boundary layer should be determined from homogeneous equations corresponding to (2.5) with homogeneous (zero) boundary conditions at $\xi = \pm 1$. Making substitution in these homogeneous equations by variable [2,5,7] $t = \xi/\varepsilon$ and putting index "b" (boundary) to all the quantities, we get the system

$$\begin{aligned} &\frac{\partial \sigma_{xxb}}{\partial t} + \frac{\partial \sigma_{xyb}}{\partial \zeta} = 0, \qquad \frac{\partial \sigma_{xyb}}{\partial t} + \frac{\partial \sigma_{yyb}}{\partial \zeta} = 0 \\ &\varepsilon^{-1} \frac{\partial U_b}{\partial t} = \frac{1}{E}(\sigma_{xxb} - \nu \sigma_{yyb}), \qquad \varepsilon^{-1} \frac{\partial V_b}{\partial \zeta} = \frac{1}{E}(\sigma_{yyb} - \nu \sigma_{xxb}) \\ &\varepsilon^{-1} \frac{\partial U_b}{\partial \zeta} + \varepsilon^{-1} \frac{\partial V_b}{\partial t} = \frac{1}{G} \sigma_{xyb} \end{aligned} \tag{2.16}$$

It is necessary to find such a solution of system (2.16), which satisfies the conditions

$$\sigma_{xyb}(t, \pm 1) = 0, \qquad \sigma_{yyb}(t, \pm 1) = 0 \tag{2.17}$$

and has fading character when removing from $x = 0$ $(t = 0)$ into the inside the rectangle-strip. This solution has the form

$$\begin{aligned} &\sigma_{ijb}(t,\zeta) = \varepsilon^{-1+s} \sigma_{ijb}^{(s)}(\zeta) \exp(-\lambda t), \qquad s = \overline{0, N} \\ &(U_b, V_b) = \varepsilon^{s} \left(u_b^{(s)}(\zeta), \text{v}_b^{(s)}(\zeta) \right) \exp(-\lambda t) \end{aligned} \tag{2.18}$$

where λ is yet the unknown number.

Substituting (2.18) into (2.16), all the unknowns may be expressed through $\sigma_{yyb}^{(s)}$:

$$\begin{aligned} &\sigma_{xxb}^{(s)} = \frac{1}{\lambda^2} \frac{d^2 \sigma_{yyb}^{(s)}}{d\zeta^2}, \qquad \sigma_{xyb}^{(s)} = \frac{1}{\lambda} \frac{d\sigma_{yyb}^{(s)}}{d\zeta} \\ &u_b^{(s)} = -\frac{1}{\lambda^3 E} \left(\frac{d^2 \sigma_{yyb}^{(s)}}{d\zeta^2} - \nu \lambda^2 \sigma_{yyb}^{(s)} \right) \\ &\text{v}_b^{(s)} = -\frac{1}{\lambda^4 E} \left(\frac{d^3 \sigma_{yyb}^{(s)}}{d\zeta^3} + (2+\nu)\lambda^2 \frac{d\sigma_{yyb}^{(s)}}{d\zeta} \right) \end{aligned} \tag{2.19}$$

which is determined from the equation

$$\frac{d^4\sigma_{yyb}^{(s)}}{d\zeta^4}+2\lambda^2\frac{d^2\sigma_{yyb}^{(s)}}{d\zeta^2}+\lambda^4\sigma_{yyb}^{(s)}=0 \tag{2.20}$$

From the first two formulae of (2.19) and conditions (2.17) follows a very important property – self equilibrium of the stresses σ_{xxb}, σ_{xyb} in the arbitrary cross-section $t=t_k$

$$\int_{-1}^{+1}\sigma_{xxb}(t,\zeta)d\zeta=0,\quad \int_{-1}^{+1}\zeta\sigma_{xxb}d\zeta=0,\quad \int_{-1}^{+1}\sigma_{xyb}d\zeta=0,\quad \forall t=t_k \tag{2.21}$$

But the displacements do not have this property (2.21), i.e.

$$\int_{-1}^{+1}U_b d\zeta\neq 0,\quad \int_{-1}^{+1}\zeta U_b d\zeta\neq 0,\quad \int_{-1}^{+1}V_b d\zeta\neq 0 \tag{2.22}$$

Having solved (2.20) and satisfied conditions (2.17) we get

$$\sigma_{yyb}^{(s)}=A_n^{(s)}F_n(\zeta),\qquad n=\overline{0,N} \tag{2.23}$$

In the symmetrical problem (tension-compression) σ_{xxb}, u_b, σ_{yyb} are even, and v_b, σ_{xyb} are odd functions from ζ, in the skew-symmetrical (bending) problem it is vice versa. We have

symmetric problem

$$F_n(\zeta)=\zeta\sin\lambda_n\zeta-\tan\lambda_n\cos\lambda_n\zeta,\qquad \sin 2\lambda_n+2\lambda_n=0 \tag{2.24}$$

skew-symmetric problem

$$F_n(\zeta)=\sin\lambda_n\zeta-\zeta\tan\lambda_n\cos\lambda_n\zeta,\qquad \sin 2\lambda_n-2\lambda_n=0 \tag{2.25}$$

Transcendental equations $\sin 2\lambda_n\pm 2\lambda_n=0$ have complex conjugate roots (except trivial $\lambda=0$), situated symmetrically relative to the axes of the coordinates. We are interested in the roots with $\mathrm{Re}\,\lambda_n>0$, providing the fading character of the solution. The values of the first five roots with $\mathrm{Re}\,\lambda_n>0$ are brought in Table 1.

Table 1

$2\lambda_n=X_n\pm iY_n$				
	$\sin 2\lambda_n+2\lambda_n=0$		$\sin 2\lambda_n-2\lambda_n=0$	
n	X_n	Y_n	X_n	Y_n
1.	4.2124	2.2507	7.4977	2.7687
2.	10.7125	3.1031	13.8999	3.3522
3.	17.0734	3.5511	20.2385	3.7168
4.	23.3984	3.8588	26.5545	3.9831
5.	29.7081	4.0937	32.8597	4.1933

Solution (2.18), (2.19), (2.23) may be transformed so that the real quantities should appeare only. For any of the stresses and displacements Q_b, admitting

$Q_{nb}^{(s)} = A_n^{(s)} \widetilde{Q}_{nb}(t,\zeta)$, where $\widetilde{Q}_{nb} = Q_{nb}(\zeta)\exp(-\lambda_n t)$, Q_{nb} is the coefficient under the arbitrary constant $A_n^{(s)}$ for the given quantity, representing $A_n^{(s)} = \frac{1}{2}(A_{1n}^{(s)} - iA_{2n}^{(s)})$, we have

$$Q_{nb}^{(s)} = \mathrm{Re}\,\widetilde{Q}_{nb} A_{1n}^{(s)} + \mathrm{Im}\,\widetilde{Q}_{nb} A_{2n}^{(s)} \tag{2.26}$$

Note that solution (2.18), (2.26) is exact solution for arbitrary s. It is known in elasticity theory as Shiff-Popkovich-Lourier homogeneous solution.

The solution of the boundary layer $I_b^{(2)}$ at $x = l$ may be obtained from the solution at $x = 0$ by formal replacement t with $t_1 = 1/\varepsilon - t$.

2.1. The connection with classical theory of beams and bars. The solution of the inner problem (2.7)-(2.13) has a direct connection with classical theory of beams bend and tension-compression of bars. In order to reveal this connection we write equations (2.12) in dimensional coordinates and displacements, according to (2.7), (2.8)

$$\begin{aligned} u &= l\varepsilon^{-2+s} u^{(s)} = u_0 + \varepsilon u_1 + \cdots + \varepsilon^s u_s + \cdots \\ \mathrm{v} &= l\varepsilon^{-3+s} \mathrm{v}^{(s)} = \mathrm{v}_0 + \varepsilon \mathrm{v}_1 + \cdots + \varepsilon^s \mathrm{v}_s + \cdots \\ u^{(s)} &= \frac{1}{l}\varepsilon^2 u_s, \qquad \mathrm{v}^{(s)} = \frac{1}{l}\varepsilon^3 \mathrm{v}_s \end{aligned} \tag{2.27}$$

Substituting values for $u^{(s)}$, $\mathrm{v}^{(s)}$ into equations (2.12) we get

$$EF\frac{d^2u_s}{dx^2} = q_{x0}^{(s)}, \qquad q_{x0}^{(s)} = 2\varepsilon^{-1} q_x^{(s)}, \qquad F = 2h\cdot 1 \tag{2.28}$$

$$EJ\frac{d^4\mathrm{v}_s}{dx^4} = q_0^{(s)}, \qquad q_0^{(s)} = 2q^{(s)}, \qquad J = \frac{2}{3}h^3\cdot 1 \tag{2.29}$$

where EF is the rigidness of the bar under tension-compression, EJ is the rigidness of the beam under bend. At $s = 0$ $q_{x0}^{(0)} = -\left(X^+ + X^-\right)$, $q^{(0)} = \left(Y^+ + Y^-\right) + h\frac{d}{dx}\left(X^+ - X^-\right)$, equation (2.28) coincides with the classical equation of bars tension-compression [8], and equation (2.29) coincides with the classical equation of beams bend [8, 9]. Moreover, the initial approach of the asymptotic solution of the inner problem contains more information, as by formulae (2.9)-(2.11), (2.13) the stresses σ_{xy}, σ_{yy}, the last of which in the classical theory is neglected at all, are calculated as well. The approaches $s \geq 1$ correct the results on classical theory, displacements and stresses, corresponding them, change along the transverse coordinate ζ nonlinearly. So, admitting the hypothesis of plane sections, approaches $s \geq 1$ of the inner problem and the boundary layer, to which the new exact solution of elasticity theory equations (homogeneous solution), which is not possible to obtain with the method of hypotheses, corresponds, are neglected.

2.2. Conjugation of the solutions of the inner problem and the boundary layer. Let at $x = 0$ conditions (2.15)*a* be given. According to (2.6) $I = I^{int} + I_b^{(1)} + I_b^{(2)}$. When satisfying the equations on the end-wall $x = 0$, the influence of $I_b^{(2)}$ is usually neglected. It equivalent to $1 + \exp(-\operatorname{Re}\lambda_1 l/h) \approx 1$, which, according to Table 1, always takes place for real beams and bars ($l \geq 10h$). We have

$$\begin{aligned} \sigma_{xx} &= \varepsilon^{-2+s}\sigma_{xx}^{(s)} + \varepsilon^{\chi-1+s}\sigma_{xxb}^{(s)} \\ \sigma_{xy} &= \varepsilon^{-1+s}\sigma_{xy}^{(s)} + \varepsilon^{\chi-1+s}\sigma_{xyb}^{(s)} \end{aligned} \tag{2.30}$$

Multiplier ε^{χ} expresses the fact that the solution of the boundary layer, as the solution of the homogeneous boundary value problem, is determined with the accuracy of the constant multiplier. χ should be determined so, that during the satisfaction of conditions (2.15) *a* contradictions wouldn't arise. It is achieved at $\chi = -1$. As a result we have

$$\begin{aligned} &\sigma_{xx}^{(s)}(x=0,\zeta) + \sigma_{xxb}^{(s)}(t=0,\zeta) = \varphi^{(s-2)} \\ &\sigma_{xy}^{(s-1)}(x=0,\zeta) + \sigma_{xyb}^{(s)}(t=0,\zeta) = \psi^{(s-2)} \\ &\varphi^{(0)} = 0, \quad \varphi^{(s)} = 0, \ s \neq 0 \quad (\varphi, \psi) \end{aligned} \tag{2.31}$$

Functions $\sigma_{xxb}^{(s)}, \sigma_{xyb}^{(s)}$ satisfy conditions (2.21). As a result we have the conditions

$$\begin{aligned} &\int_{-1}^{+1} \sigma_{xx}^{(s)}(0,\zeta)d\zeta = \int_{-1}^{+1} \varphi^{(s-2)}d\zeta, \quad \int_{-1}^{+1} \zeta\sigma_{xx}^{(s)}(0,\zeta)d\zeta = \int_{-1}^{+1} \zeta\varphi^{(s-2)}d\zeta \\ &\int_{-1}^{+1} \sigma_{xy}^{(s)}(0,\zeta)d\zeta = \int_{-1}^{+1} \psi^{(s-1)}d\zeta \end{aligned} \tag{2.32}$$

From the three conditions (2.32) constants of the solution of the inner problem $C_1^{(s)}$, $C_3^{(s)}$, $C_4^{(s)}$ are determined. It is interesting that in (2.32) the conditions turned out to be as many as the unknown constants are in the solution of the inner problem, which indicates the presence of the inner harmony in elasticity theory. Coming back to (2.31), where $\sigma_{xx}^{(s)}$, $\sigma_{xy}^{(s)}$ are already known functions, for determining the constants in the solution of the boundary layer we obtain the conditions

$$\begin{aligned} \sigma_{xxb}^{(s)}(t=0,\zeta) &= \varphi^{(s-2)} - \sigma_{xx}^{(s)}(0,\zeta) \\ \sigma_{xyb}^{(s)}(t=0,\zeta) &= \psi^{(s-2)} - \sigma_{xy}^{(s-1)}(0,\zeta) \end{aligned} \tag{2.33}$$

or

$$\begin{aligned} \operatorname{Re}\tilde{\sigma}_{xxb}(0,\zeta)A_{1n}^{(s)} + \operatorname{Im}\tilde{\sigma}_{xxb}(0,\zeta)A_{2n}^{(s)} &= \overline{\varphi}^{(s)} \\ \operatorname{Re}\tilde{\sigma}_{xyb}(0,\zeta)A_{1n}^{(s)} + \operatorname{Im}\tilde{\sigma}_{xyb}(0,\zeta)A_{2n}^{(s)} &= \overline{\psi}^{(s)} \end{aligned} \tag{2.34}$$

where $\overline{\varphi}^{(s)}$, $\overline{\psi}^{(s)}$ are the right parts of conditions (2.33). For the calculation of the values of $A_{1n}^{(s)}$ and $A_{2n}^{(s)}$ from system (2.34), collocation method, Fourier method, the method of least squares etc., may be used.

2.3. Connection with Saint-Venant principle. From conditions (2.32), (2.33) a very important fact follows – if at $x=0$ conditions (2.15)a relative to the stresses are given, then the solution of the inner problem takes non-self-balanced part of the load on itself, and the boundary layer by virtue of (2.33) takes the self-balanced part of the end-wall load on itself. From this purely mathematical fact follows that if the beam (bar) is loaded by the end-wall load, then the same inner stress-strain state will statically correspond to the equivalent loads. This is the very principle of Saint-Venant. So, justification of this principle in case of the first boundary value problem of elasticity theory for a rectangle-strip is mathematically proved. Let's show what was above said in some examples. Let in (2.15)a 1) $\varphi=\frac{3}{2}p(1-\zeta^2),\ \psi=0$, 2) $\varphi=2p|\zeta|,\ \psi=0$, and at $x=l$ $\sigma_{xx}=p,\ \sigma_{xy}=0$. In both cases $\sigma_{xx}=p,\ \sigma_{xy}=\sigma_{yy}=0$ is the solution of the inner problem. The difference will be in the solutions for the boundary layer. We have (fig. 3), (fig. 4)

1)

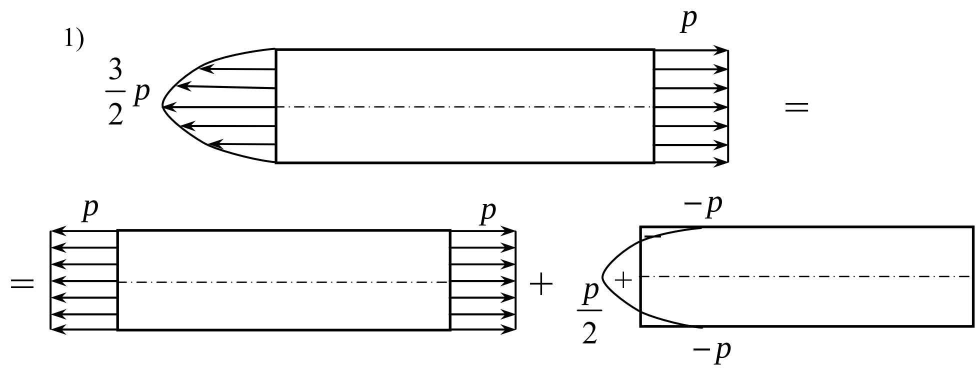

Fig. 3

The fading solution when removing from the end-wall $x=0$, according to (2.33) will correspond to functions $\overline{\varphi}=\frac{p}{2}(1-\zeta^2),\ \overline{\psi}=0$. The graphs of the stresses are brought in [5].

2)

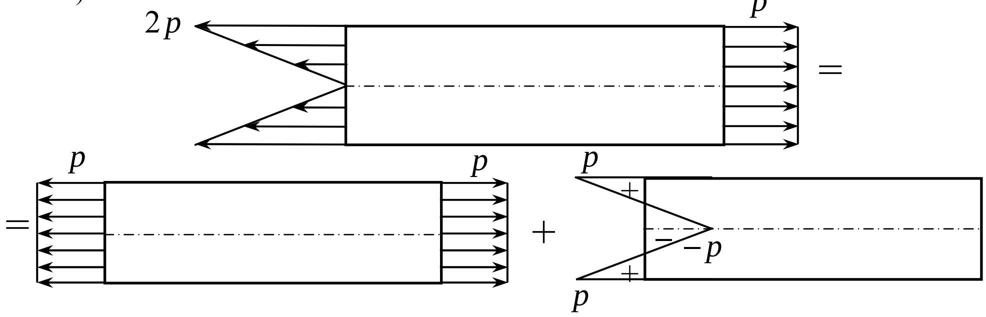

Fig. 4

The fading solution corresponds to functions $\overline{\varphi} = -2p\left(\frac{1}{2} - |\zeta|\right)$, $\overline{\psi} = 0$.

In analogous way one and the same solution of beam pure bend problem ($\sigma_{xx} = -p\zeta$, $\sigma_{xy} = 0$, $\sigma_{yy} = 0$ at $x = l$) corresponds to conditions 1) $\varphi(\zeta) = \frac{5}{3}p\zeta^3$, $\psi(\zeta) = 0$, 2) $\varphi(\zeta) = \frac{\pi^2}{12}p\sin\frac{\pi}{2}\zeta$, $\psi(\zeta) = 0$. The initial approach already gives the exact solution of the inner problem: $\sigma_{xx}^{int} = -\frac{M}{J}y$, $\sigma_{xy} = \sigma_{yy} = 0$, where M is the bending moment, J is the moment of inertia of the cross-section, which coincides with the well-known elementary solution of elasticity theory. The fading solutions correspond to the end-wall values of stresses 1) $\overline{\varphi} = \frac{p}{3}(5\zeta^3 - 3\zeta)$, $\overline{\psi} = 0$, 2) $\overline{\varphi} = \frac{p}{3}\left(\frac{\pi^2}{4}\sin\frac{\pi}{2}\zeta - 3\zeta\right)$, $\overline{\psi} = 0$, the graphs of the stresses are brought in [5].

If at $x = 0$ the conditions relative to displacements (2.15)b are given, by virtue of (2.22) the conjugation of the solutions of the inner problem and the boundary layer should be conducted in another way, for example, by the method of least squares. It is obvious that Saint-Venant principle must not be formally spread over the displacements.

3. Asymptotic solutions of the boundary value problem for anisotropic beams and plates. The asymptotic method of the boundary value problem for an isotropic strip without any difficulty is spread for a strip-rectangle having in its plane general anisotropy, and for a layered strip-beam, as well. In the first case we have the following state correlations

$$\begin{aligned}
\frac{\partial u}{\partial x} &= a_{11}\sigma_{xx} + a_{12}\sigma_{yy} + a_{16}\sigma_{xy} + \alpha_{11}\theta \\
\frac{\partial \mathrm{v}}{\partial y} &= a_{12}\sigma_{xx} + a_{22}\sigma_{yy} + a_{26}\sigma_{xy} + \alpha_{22}\theta \\
\frac{\partial u}{\partial y} + \frac{\partial \mathrm{v}}{\partial x} &= a_{16}\sigma_{xx} + a_{26}\sigma_{yy} + a_{66}\sigma_{xy} + \alpha_{12}\theta
\end{aligned} \tag{3.1}$$

Transforming according to (2.4) the equations of equilibrium (2.1) and correlations (3.1), we obtain singularly perturbed by small parameter ε system, the solution of which will have the form (2.6)-(2.8). The solution of the inner problem is again expressed by the analogous formulae through functions $u^{(s)}$, $\mathrm{v}^{(s)}$, which are determined from equations

$$\frac{2}{a_{11}}\frac{d^2u^{(s)}}{d\xi^2} = q_x^{(s)}, \quad q_x^{(s)} = -\left[X^{+(s)} + X^{-(s)} - \sigma_{xy*}^{(s)}(\xi,1) + \sigma_{xy*}^{(s)}(\xi,-1)\right]$$

$$\frac{2}{3a_{11}}\frac{d^4\mathrm{v}^{(s)}}{d\xi^4} = q^{(s)}, \quad a_{11} = \frac{1}{E_1} \tag{3.2}$$

$$q^{(s)} = Y^{+(s)} + Y^{-(s)} - \sigma^{(s)}_{y*}(\xi,1) + \sigma^{(s)}_{y*}(\xi,-1) +$$
$$+ \frac{d}{d\xi}\left[X^{+(s)} - X^{-(s)} - \sigma^{(s)}_{xy*}(\xi,1) - \sigma^{(s)}_{xy*}(\xi,-1) \right]$$

The solution of the boundary layer have the form (2.18) and is determined by formulae

$$\sigma^{(s)}_{yyb} = A^{(s)}_n F_n(\zeta), \quad \sigma^{(s)}_{xyb} = A^{(s)}_n \lambda_n^{-1} F_n'(\zeta), \quad \sigma^{(s)}_{xxb} = A^{(s)}_n \lambda_n^{-2} F_n''(\zeta)$$
$$u^{(s)}_b = -A^{(s)}_n \left(a_{11}\lambda_n^{-3} F_n'' + a_{16}\lambda_n^{-2} F_n' + a_{12}\lambda_n^{-1} F_n \right) \tag{3.3}$$
$$\mathrm{v}^{(s)}_b = -A^{(s)}_n \left(a_{11}\lambda_n^{-4} F_n''' + 2a_{16}\lambda_n^{-3} F_n'' + (a_{12}+a_{66})\lambda_n^{-2} F_n' + a_{26}\lambda_n^{-1} F_n \right)$$

where functions F_n are determined from the boundary value problem

$$a_{11}F_n^{IV} + 2a_{16}\lambda_n F_n''' + (2a_{12}+a_{66})\lambda_n^2 F_n'' + 2a_{26}\lambda_n^3 F_n' + a_{22}\lambda_n^4 F_n = 0$$
$$F_n(\zeta = \pm 1) = 0, \qquad F_n'(\zeta = \pm 1) = 0 \tag{3.4}$$

From (3.3), (3.4) the self-balance of stresses σ_{xxb}, σ_{xyb} again follows in the arbitrary cross-section $t = t_k$, i.e. Saint-Venant principle takes place for the strip-beam having a general anisotropy in the plane of the cross-section, as well.

The conjugation of the inner problem and boundary layer solutions is realized in the similar way.

In case of a layered strip-beam it is necessary to assign all the values entering into the equations of the equilibrium (2.1) and elasticity correlations (3.1) by index $"k"$ (k is the layer number) and to seek the solution of the transformed system in the form of (2.6)-(2.8). As a result for the values $"k"$ of that layer we have

$$V^{(k,s)} = \mathrm{v}^{(k,s)}(\xi) + \mathrm{v}^{(k,s)}_*(\xi,\zeta)$$
$$U^{(k,s)} = -\frac{d\mathrm{v}^{(k,s)}}{d\xi}\zeta + u^{(k,s)}(\xi) + u^{(k,s)}_*(\xi,\zeta)$$
$$\sigma^{(k,s)}_{xx} = \frac{1}{a^{(k)}_{11}}\frac{du^{(k,s)}}{d\xi} - \frac{1}{a^{(k)}_{11}}\frac{d^2\mathrm{v}^{(k,s)}}{d\xi^2} + \sigma^{(k,s)}_{x*}(\xi,\zeta) \tag{3.5}$$
$$\sigma^{(k,s)}_{xy} = \frac{1}{a^{(k)}_{11}}\frac{d^3\mathrm{v}^{(k,s)}}{d\xi^3}\frac{\zeta^2}{2} - \frac{1}{a^{(k)}_{11}}\frac{d^2u^{(k,s)}}{d\xi^2}\zeta + \sigma^{(k,s)}_{xy0}(\xi) + \sigma^{(k,s)}_{xy*}(\xi,\zeta)$$
$$\sigma^{(k,s)}_{yy} = -\frac{1}{a^{(k)}_{11}}\frac{d^4\mathrm{v}^{(k,s)}}{d\xi^4}\frac{\zeta^3}{6} + \frac{1}{a^{(k)}_{11}}\frac{d^3u^{(k,s)}}{d\xi^3}\frac{\zeta^2}{2} - \frac{d\sigma^{(k,s)}_{xy0}}{d\xi}\zeta + \sigma^{(k,s)}_{yy0}(\xi) + \sigma^{(k,s)}_{y*}(\xi,\zeta)$$

It is not difficult to write out the expressions $Q^{(k,s)}_*$, which are the known functions. If the packet consists of $n+m$ layers, where n is the quantity of the layers stood above, and m is lower the axis $O\xi$, satisfying the boundary conditions on the facial surfaces of the packet and the conditions of full contact among the layers (continuity of the displacements and the stresses σ_{xy}, σ_{yy}), all the values may be expressed through the displacement of n^{th} layer, and for them obtain the equations [10]

$$
\begin{aligned}
&C\frac{d^2u^{(n,s)}}{d\xi^2}+K\frac{d^3\mathrm{v}^{(n,s)}}{d\xi^3}=p^{(s)}\\
&D\frac{d^4\mathrm{v}^{(n,s)}}{d\xi^4}+K\frac{d^3u^{(n,s)}}{d\xi^3}=q^{(s)}
\end{aligned}
\tag{3.6}
$$

where

$$
\begin{aligned}
&C=\sum_{k=1}^{n}\frac{1}{a_{11}^{(k)}}(\zeta_k-\zeta_{k-1})-\sum_{k=1}^{m}\frac{1}{a_{11}^{(-k)}}(\zeta_{-k}-\zeta_{-k+1})\\
&D=\frac{1}{3}\sum_{k=1}^{n}\frac{1}{a_{11}^{(k)}}\left(\zeta_k^3-\zeta_{k-1}^3\right)-\frac{1}{3}\sum_{k=1}^{m}\frac{1}{a_{11}^{(-k)}}\left(\zeta_{-k}^3-\zeta_{-k+1}^3\right)\\
&K=-\frac{1}{2}\sum_{k=1}^{n}\frac{1}{a_{11}^{(k)}}\left(\zeta_k^2-\zeta_{k-1}^2\right)+\frac{1}{2}\sum_{k=1}^{m}\frac{1}{a_{11}^{(-k)}}\left(\zeta_{-k}^2-\zeta_{-k+1}^2\right)\\
&\zeta_k=\frac{1}{h}\sum_{j=1}^{k}h_j\quad(k=1,2,\cdots,n),\quad \zeta_{-k}=-\frac{1}{h}\sum_{j=1}^{k}h_{-j}\quad(k=1,2,\cdots,m)\\
&h=\sum_{j=1}^{n}h_j+\sum_{j=1}^{m}h_{-j}
\end{aligned}
\tag{3.7}
$$

Stiffness C and D are positive, and the position of the axis $O\xi$ can be chosen so, that $K=0$. Then the first equation of (3.6) will correspond to tension-compression of the layered bar, and the second will correspond to the bend of the layered beam. At $s=0$ system (3.6) coincides with the classical system by the hypothesis of plane sections [11].

In order to solve space problem for plates $D=\{(x,y,z):0\le x\le a,\ 0\le y\le b,\ -h\le z\le +h\}$ having common anisotropy (21 elasticity constants) we input dimensionless coordinates $x=l\xi$, $y=l\eta$, $z=h\zeta$ ($l=\min(a,b),\ h<<l$) and dimensionless displacements $U=u/l$, $V=\mathrm{v}/l$, $W=w/l$. The corresponding system of thermoelasticity equations

$$
\begin{aligned}
&\frac{\partial\sigma_{xx}}{\partial\xi}+\frac{\partial\sigma_{xy}}{\partial\eta}+\varepsilon^{-1}\frac{\partial\sigma_{xz}}{\partial\zeta}+lF_x(\xi,\eta,\zeta)=0\qquad(x,y,z;\ \xi,\eta,\zeta)\\
&\frac{\partial U}{\partial\xi}=a_{11}\sigma_{xx}+a_{12}\sigma_{yy}+a_{13}\sigma_{zz}+a_{14}\sigma_{yz}+a_{15}\sigma_{xz}+a_{16}\sigma_{xy}+\alpha_{11}\theta\\
&\frac{\partial V}{\partial\eta}=a_{12}\sigma_{xx}+a_{22}\sigma_{yy}+a_{23}\sigma_{zz}+a_{24}\sigma_{yz}+a_{25}\sigma_{xz}+a_{26}\sigma_{xy}+\alpha_{22}\theta\\
&\varepsilon^{-1}\frac{\partial W}{\partial\zeta}=a_{13}\sigma_{xx}+a_{23}\sigma_{yy}+a_{33}\sigma_{zz}+a_{34}\sigma_{yz}+a_{35}\sigma_{xz}+a_{36}\sigma_{xy}+\alpha_{33}\theta\\
&\varepsilon^{-1}\frac{\partial V}{\partial\zeta}+\frac{\partial W}{\partial\eta}=a_{14}\sigma_{xx}+a_{24}\sigma_{yy}+a_{34}\sigma_{zz}+a_{44}\sigma_{yz}+a_{45}\sigma_{xz}+a_{46}\sigma_{xy}+\alpha_{23}\theta
\end{aligned}
\tag{3.8}
$$

$$\varepsilon^{-1}\frac{\partial U}{\partial \zeta}+\frac{\partial W}{\partial \xi}=a_{15}\sigma_{xx}+a_{25}\sigma_{yy}+a_{35}\sigma_{zz}+a_{45}\sigma_{yz}+a_{55}\sigma_{xz}+a_{56}\sigma_{xy}+\alpha_{13}\theta$$

$$\frac{\partial V}{\partial \xi}+\frac{\partial U}{\partial \eta}=a_{16}\sigma_{xx}+a_{26}\sigma_{yy}+a_{36}\sigma_{zz}+a_{46}\sigma_{yz}+a_{56}\sigma_{xz}+a_{66}\sigma_{xy}+\alpha_{12}\theta$$

is again singularly perturbed by small parameter $\varepsilon = h/l$. The solution of system (3.8) has the form of (2.6), for the inner problem we have

$$q=-2 \text{ for } \sigma_{xx},\ \sigma_{xy},\ \sigma_{yy},\ u,\ \mathrm{v};\quad q=-1 \text{ for } \sigma_{xz},\ \sigma_{yz};$$

$$q=0 \text{ for } \sigma_{zz};\quad q=-3 \text{ for } w \tag{3.9}$$

Substituting (3.9) into (3.8) we obtain a system which permits integration by ζ and satisfaction of the conditions at $\zeta = \pm 1$. As a result we have

$$W^{(s)}=w^{(s)}(\xi,\eta)+w_*^{(s)}(\xi,\eta,\zeta)$$

$$U^{(s)}=-\zeta\frac{\partial w^{(s)}}{\partial \xi}+u^{(s)}(\xi,\eta)+u_*^{(s)}(\xi,\eta,\zeta),\quad (u,\mathrm{v};\ \xi,\eta)$$

$$\sigma_{xx}^{(s)}=\zeta\tau_{x1}^{(s)}+\tau_{x0}^{(s)}+\sigma_{x*}^{(s)}(\xi,\eta,\zeta),\quad (x,y) \tag{3.10}$$

$$\sigma_{xy}^{(s)}=\zeta\tau_{xy1}^{(s)}+\tau_{xy0}^{(s)}+\sigma_{xy*}^{(s)}$$

$$\sigma_{xz}^{(s)}=\frac{1}{2}\zeta^2\tau_{xz2}^{(s)}+\zeta\tau_{xz1}^{(s)}+\tau_{xz0}^{(s)}+\sigma_{xz*}^{(s)},\quad (x,y)$$

$$\sigma_{z}^{(s)}=\frac{1}{6}\zeta^3\tau_{z3}^{(s)}+\frac{1}{2}\zeta^2\tau_{z2}^{(s)}+\zeta\tau_{z1}^{(s)}+\tau_{z0}^{(s)}+\sigma_{z*}^{(s)}$$

Functions $\tau^{(s)}$ are expressed through $u^{(s)}$, $\mathrm{v}^{(s)}$, $w^{(s)}$, which are determined from the equations [5]

$$l_{11}u^{(s)}+l_{12}\mathrm{v}^{(s)}=p_1^{(s)},\qquad l_{12}u^{(s)}+l_{22}\mathrm{v}^{(s)}=p_2^{(s)} \tag{3.11}$$

$$\begin{aligned} &B_{11}\frac{\partial^4 w^{(s)}}{\partial \xi^4}+4B_{16}\frac{\partial^4 w^{(s)}}{\partial \xi^3\partial \eta}+2(B_{12}+2B_{66})\frac{\partial^4 w^{(s)}}{\partial \xi^2\partial \eta^2}\\ &\quad +4B_{26}\frac{\partial^4 w^{(s)}}{\partial \xi\partial \eta^3}+B_{22}\frac{\partial^4 w^{(s)}}{\partial \eta^4}=q^{(s)}\end{aligned} \tag{3.12}$$

where

$$l_{11}=B_{11}\frac{\partial^2}{\partial \xi^2}+B_{66}\frac{\partial^2}{\partial \eta^2}+2B_{16}\frac{\partial^2}{\partial \xi\partial \eta},\quad (1,2;\ \xi,\eta)$$

$$l_{12}=B_{16}\frac{\partial^2}{\partial \xi^2}+(B_{12}+B_{66})\frac{\partial^2}{\partial \xi\partial \eta}+B_{26}\frac{\partial^2}{\partial \eta^2}$$

$$B_{11}=\left(a_{22}a_{66}-a_{26}^2\right)/\Omega,\qquad B_{22}=\left(a_{11}a_{66}-a_{16}^2\right)/\Omega \tag{3.13}$$

$$B_{12} = (a_{16}a_{26} - a_{12}a_{66})/\Omega, \qquad B_{66} = \left(a_{11}a_{22} - a_{12}^2\right)/\Omega$$
$$B_{16} = (a_{12}a_{26} - a_{22}a_{16})/\Omega, \qquad B_{26} = (a_{12}a_{16} - a_{11}a_{26})/\Omega$$
$$\Omega = \left(a_{11}a_{22} - a_{12}^2\right)a_{66} + 2a_{12}a_{16}a_{26} - a_{11}a_{26}^2 - a_{22}a_{16}^2$$

Equations (3.11), written in dimensional coordinates, coincide at $s = 0$ with classical equations of the generalized plane problem, and equation (3.12) coincides with the classical equation of the plate bend, which has plane of the elastic symmetry [11]. At $s > 0$ the right parts of these equations (load members) change, where the elasticity coefficients of mutual influence, characterizing general anisotropy, enter too. Under the general anisotropy $p_1^{(1)} \neq 0$, $p_2^{(1)} \neq 0$, $q^{(1)} \neq 0$ and the error of the classical theory makes up $O(\varepsilon)$, when, as for isotropic, orthotropic plates and plates having plane elastic symmetry under the tempered anisotropy error is $O(\varepsilon^2)$, and under strong anisotropy these estimates sharply change into the worse side.

Classical theory of plates does not take into account the boundary layer. The solution of the boundary layer localized in the vicinity of the end $\xi = 0$ is sought in the form of

$$I_b = \varepsilon^{\chi+s} I_b^{(s)}(\eta,\zeta)\exp(-\lambda t), \quad t = \xi/\varepsilon \tag{3.14}$$

where $\chi = 0$ for u_b, v_b, w_b; $\chi = -1$ for σ_{ijb}. In general case all the values of the boundary layer are expressed through the functions $\sigma_{zzb}^{(s)}$, $\sigma_{yzb}^{(s)}$, which are determined from the equations [5]

$$\begin{aligned} L_1\sigma_{zzb}^{(s)} + L_2\sigma_{yzb}^{(s)} &= R_1^{(s-1)} \\ L_2\sigma_{zzb}^{(s)} + L_3\sigma_{yzb}^{(s)} &= R_2^{(s-1)} \end{aligned} \tag{3.15}$$

under the boundary conditions

$$\sigma_{zzb}^{(s)}(\zeta = \pm 1) = 0, \quad \sigma_{xzb}^{(s)}(\zeta = \pm 1) = 0, \quad \sigma_{yzb}^{(s)}(\zeta = \pm 1) = 0 \tag{3.16}$$

where $R_1^{(s-1)}$, $R_2^{(s-1)}$ are known functions, $R_1^{(0)} = R_2^{(0)} = 0$, and

$$\begin{aligned} L_1 &= A_{11}\frac{\partial^4}{\partial\zeta^4} + 2A_{15}\lambda\frac{\partial^3}{\partial\zeta^3} + (2A_{13} + A_{35})\lambda^2\frac{\partial^2}{\partial\zeta^2} + 2A_{45}\lambda^3\frac{\partial}{\partial\zeta} + A_{43}\lambda^4 \\ L_2 &= A_{16}\lambda\frac{\partial^3}{\partial\zeta^3} + (A_{14} + A_{25})\lambda^2\frac{\partial^2}{\partial\zeta^2} + (A_{23} + A_{34})\lambda^3\frac{\partial}{\partial\zeta} + A_{44}\lambda^4 \\ L_3 &= A_{26}\lambda^2\frac{\partial^2}{\partial\zeta^2} + 2A_{24}\lambda^3\frac{\partial}{\partial\zeta} + A_{41}\lambda^4 \end{aligned} \tag{3.17}$$

At $s = 0$ admitting $\sigma_{zzb}^{(0)} = L_3\phi$, $\sigma_{yzb}^{(0)} = -L_2\phi$, the solution of the problem (3.15), (3.16) is reduced to the boundary value problem

$$\begin{aligned} &\left(L_1L_3 - L_2^2\right)\phi = 0 \\ &(L_3\phi)_{\zeta=\pm1} = 0, \quad (L_3\phi')_{\zeta=\pm1} = 0, \quad (L_2\phi)_{\zeta=\pm1} = 0 \end{aligned} \tag{3.18}$$

The operator of the problem (3.18) is ordinary differential of the sixth order. The system from six homogeneous algebraic equations corresponds to the boundary value problem (3.18), equalizing the determinant of this system to zero we obtain an equation for λ. $\mathrm{Re}\lambda_n > 0$ of this equation will characterize the rate of decrease of the boundary layer quantities. For isotropic and orthotropic plates $L_2 \equiv 0$ and the boundary value problem (3.15), (3.16) splits at $s = 0$ into two independent problems

$$\begin{aligned} &L_1\sigma_{zzb}^{(0)} = 0, \quad \sigma_{zzb}^{(0)}(\zeta = \pm 1) = 0, \quad \left(\sigma_{zzb}^{(0)}\right)'_{\zeta=\pm 1} = 0 \\ &L_3\sigma_{yzb}^{(0)} = 0, \quad \sigma_{yzb}^{(0)}(\zeta = \pm 1) = 0 \end{aligned} \tag{3.19}$$

to which plane and antiplane boundary layers correspond. In the first case λ_n are complex conjugate, in the second case they are real, naturally, their values depend on the values of the elasticity constants. For the plates from composite materials the values λ_n are brought in [5].Conjugation of the solutions of the inner problem and the boundary layer is realized by the similar way, described above.

In case of shells the structure of solution (2.6) remains unchangeable, yet in the inner problem asymptotics differs from (3.9). Iteration processes, corresponding to momentless and moment stress-strain state of shells, are built, the conjugation of the inner problem and boundary layer solutions is realized as for plates [5].

II. Nonclassical boundary value problems of beams, plates and shells

Classical and precise theories of beams, plates and shells consider only one class of problems – on the facial surfaces of the thin body the values of the corresponding stresses tensor components are given, Meanwhile, in fundamental construction, seismic construction, aero-vessel construction and other areas classes of problems, when on the facial surfaces of the thin body other conditions – displacement vector, mixed conditions are given, arise. This class of problems is accepted to call nonclassical, in order to differ from the second and mixed problems of classical theory (similar conditions are given on the lateral area), though from the position of elasticity theory these problems are fully classical. Using the cited in chapter 2,3 correlations or correlations of classical theory, by direct check it is possible to verify, that they can't satisfy the conditions of the second and mixed boundary value problems of elasticity theory on the facial surfaces, which means inapplicability of Bernoulli-Kirchhoff-Love hypothesis for the solution of this class of problems. The asymptotic method permits us to find the solutions of these boundary value problems for beams-strips, plates and shells, in addition to anisotropic and layered ones without using any hypothesis [5,13].

1. In case of anisotropic rectangular beams-strips it is required to find the equations solution of plane problems of thermoelasticity in the area $D = \{(x, y) : 0 \le x \le \ell, -h \le y \le h, h << \ell\}$ under the conditions

$$u(x,-h) = u^-(x), \ \mathrm{v}(x,-h) = \mathrm{v}^-(x), \text{ particularly } u^- = \mathrm{v}^- = 0 \tag{1.1}$$

and one of the groups of conditions at $y = h$

$$u(x,h) = u^+(x), \ \mathrm{v}(x,h) = \mathrm{v}^+(x) \tag{1.2}$$

$$\sigma_{xy}(x,h) = \sigma_{xy}^+(x), \ \sigma_{yy}(x,h) = \sigma_{yy}^+(x) \tag{1.3}$$

$$\mathrm{v}(x,h) = \mathrm{v}^+(x), \ \ \sigma_{xy}(x,y) = \sigma_{xy}^+(x) \tag{1.4}$$

$$u(x,h) = u^+(x), \ \sigma_{yy}(x,h) = \sigma_{yy}^+(x) \tag{1.5}$$

and conditions at $x = 0, \ell$. Passing to dimensionless coordinates $\xi = x/\ell,\ \zeta = y/h$, and dimensionless displacement $U = u/\ell, V = \mathrm{v}/\ell$, the equations of elasticity theory will be singularly perturbed by small parameter $\varepsilon = h/\ell$. The solution of this system again has the form (I.2.6), but the asymptotics of the inner problem permitting to find the solutions corresponding to conditions (1.1)-(1.5) is different:

$$I^{\text{int}} = \varepsilon^{q+s} I^{(s)}(\xi,\zeta)$$
$$q = -1 \quad for \quad \sigma_{xx},\ \sigma_{xy},\ \sigma_{yy}; \quad q = 0 \quad for \quad U, V \tag{1.6}$$

which corresponds to the equations

$$\frac{\partial \sigma_{xx}^{(s-1)}}{\partial \xi} + \frac{\partial \sigma_{xy}^{(s)}}{\partial \zeta} + F_x^{(s)}(\xi,\zeta) = 0, \quad \frac{\partial \sigma_{xy}^{(s)}}{\partial \xi} + \frac{\partial \sigma_{yy}^{(s)}}{\partial \zeta} + F_y^{(s)}(\xi,\zeta) = 0$$
$$\frac{\partial U^{(s-1)}}{\partial \xi} = a_{11}\sigma_{xx}^{(s)} + a_{12}\sigma_{yy}^{(s)} + a_{16}\sigma_{xy}^{(s)} + \alpha_{11}\theta^{(s)}(\xi,\zeta)$$
$$\frac{\partial U^{(s)}}{\partial \zeta} + \frac{\partial V^{(s-1)}}{\partial \xi} = a_{16}\sigma_{xx}^{(s)} + a_{26}\sigma_{yy}^{(s)} + a_{66}\sigma_{yy}^{(s)} + \alpha_{12}\theta^{(s)}(\xi,\zeta) \tag{1.7}$$
$$\frac{\partial V^{(s)}}{\partial \zeta} = a_{12}\sigma_{xx}^{(s)} + a_{22}\sigma_{yy}^{(s)} + a_{26}\sigma_{xy}^{(s)} + \alpha_{22}\theta^{(s)}(\xi,\zeta)$$
$$F_x^{(0)} = \varepsilon^2 \ell F_x, \quad F_x^{(s)} = 0, \ \ s \neq 0 \ \ (x,y), \ \theta^{(0)} = \varepsilon\theta, \ \ \theta^{(s)} = 0, \ s \neq 0$$

System (1.7) assumes integration by ζ :

$$\sigma_{xy}^{(s)} = \sigma_{xy0}^{(s)}(\xi) + \sigma_{xy*}^{(s)}(\xi,\zeta), \quad \sigma_{yy}^{(s)} = \sigma_{yy0}^{(s)}(\xi) + \sigma_{y*}^{(s)}(\xi,\zeta)$$
$$\sigma_{xx}^{(s)} = -(a_{12}\sigma_{yy0}^{(s)} + a_{16}\sigma_{xy0}^{(s)})a_{11}^{-1} + \sigma_{x*}^{(s)}(\xi,\zeta) \tag{1.8}$$
$$U^{(s)} = (A_{16}\sigma_{yy0}^{(s)} + A_{66}\sigma_{xy0}^{(s)})\zeta + u^{(s)}(\xi) + u_*^{(s)}(\xi,\zeta)$$
$$V^{(s)} = (A_{11}\sigma_{yy0}^{(s)} + A_{16}\sigma_{xy0}^{(s)})\zeta + \mathrm{v}^{(s)}(\xi) + \mathrm{v}_*^{(s)}(\xi,\zeta)$$

where

$$\sigma_{xy*}^{(s)} = -\int_0^\zeta \left[F_x^{(s)} + \frac{\partial \sigma_{xx}^{(s-1)}}{\partial \xi} \right] d\zeta, \quad \sigma_{y*}^{(s)} = -\int_0^\zeta \left[F_y^{(s)} + \frac{\partial \sigma_{xy}^{(s-1)}}{\partial \xi} \right] d\zeta$$
$$\sigma_{x*}^{(s)} = -[a_{12}\sigma_{y*}^{(s)} + a_{16}\sigma_{xy*}^{(s)} + \alpha_{11}\theta^{(s)} - \frac{\partial U^{(s-1)}}{\partial \xi}]a_{11}^{-1}$$
$$u_*^{(s)} = \int_0^\zeta \left[a_{16}\sigma_{x*}^{(s)} + a_{26}\sigma_{y*}^{(s)} + a_{66}\sigma_{xy*}^{(s)} + \alpha_{12}\theta^{(s)} - \frac{\partial V^{(s-1)}}{\partial \xi} \right] d\zeta \tag{1.9}$$
$$\mathrm{v}_*^{(s)} = \int_0^\zeta \left[a_{12}\sigma_{x*}^{(s)} + a_{22}\sigma_{y*}^{(s)} + a_{26}\sigma_{xy*}^{(s)} + \alpha_{22}\theta^{(s)} \right] d\zeta$$
$$A_{11} = (a_{11}a_{22} - a_{12}^2)a_{11}^{-1}, \quad A_{16} = (a_{11}a_{26} - a_{12}a_{16})a_{11}^{-1}$$
$$A_{66} = (a_{11}a_{66} - a_{16}^2)a_{11}^{-1}$$

Unknown functions $\sigma_{xy0}^{(s)}, \sigma_{y0}^{(s)}, u^{(s)}, \mathrm{v}^{(s)}$, which are unambiguously determined from conditions (1.1)-(1.5), enter the solution of the inner problem (1.6), (1.8). For example, for conditions (1.1), (1.2) we have

$$\sigma_{xy0}^{(s)} = (A_{11}\varphi^{(s)} - A_{16} f^{(s)})\Omega^{-1},\ \sigma_{yy0}^{(s)} = (A_{66} f^{(s)} - A_{16}\varphi^{(s)})\Omega^{-1}$$

$$\begin{aligned} u^{(s)}(\xi) &= \frac{1}{2}(u^{+(s)} + u^{-(s)}) - \frac{1}{2}(u_*^{(s)}(\xi,1) + u_*^{(s)}(\xi,-1)) \\ \mathrm{v}^{(s)}(\xi) &= \frac{1}{2}(\mathrm{v}^{+(s)} + \mathrm{v}^{-(s)}) - \frac{1}{2}(\mathrm{v}_*^{(s)}(\xi,1) + \mathrm{v}_*^{(s)}(\xi,-1)) \\ \varphi^{(s)}(\xi) &= \frac{1}{2}(u^{+(s)} - u^{-(s)}) - \frac{1}{2}(u_*^{(s)}(\xi,1) - u_*^{(s)}(\xi,-1)) \\ f^{(s)}(\xi) &= \frac{1}{2}(\mathrm{v}^{+(s)} - \mathrm{v}^{-(s)}) - \frac{1}{2}(\mathrm{v}_*^{(s)}(\xi,1) - \mathrm{v}_*^{(s)}(\xi,-1)) \end{aligned} \tag{1.10}$$

$$\Omega = A_{11}A_{66} - A_{16}^2,\ u^{\pm(0)} = u^{\pm}/\ell,\ u^{\pm(s)} = 0,\ s \neq 0,\ (u, \mathrm{v})$$

In the problem corresponding to conditions (1.1), (1.3) we have

$$\begin{aligned} &\sigma_{xy0}^{(s)} = \sigma_{xy}^{+(s)} - \sigma_{xy*}^{(s)}(\xi,1), \quad \sigma_{yy0}^{(s)} = \sigma_{yy}^{+(s)} - \sigma_{y*}^{(s)}(\xi,1) \\ &u^{(s)}(\xi) = A_{16}\sigma_{yy0}^{(s)} + A_{66}\sigma_{xy0}^{(s)} + u^{-(s)} - u_*^{(s)}(\xi,-1) \\ &\mathrm{v}^{(s)} = A_{11}\sigma_{yy0}^{(s)} + A_{16}\sigma_{xy0}^{(s)} + \mathrm{v}^{-(s)}(\xi) - \mathrm{v}_*^{(s)}(\xi,-1) \\ &\sigma_{xy}^{+(0)} = \varepsilon\sigma_{xy}^{+},\ \ \sigma_{yy}^{+(0)} = \varepsilon\sigma_{yy}^{+},\ \ \sigma_{xy}^{+(s)} = \sigma_{yy}^{+(s)} = 0, \quad s \neq 0 \end{aligned} \tag{1.11}$$

It is not difficult to extract the values of these functions under conditions (1.1), (1.4); (1.1), (1.5). So, unlike the first boundary value problem, the solution of the inner problem, under the different conditions on the facial surfaces $y = \pm h$, is fully determined as a result of satisfaction of these conditions. If the boundary functions are polynomials from variable ξ, the iteration process breaks and the solution of the inner problem becomes mathematically exact. For example, at $u^{+} = \mathrm{const},\ \mathrm{v}^{+} = \mathrm{const}$ from (1.6), (1.10) follows

$$\begin{aligned} &\sigma_{xy} = (A_{11}u^{+} - A_{16}\mathrm{v}^{+})/2h\Omega, \ \ \sigma_{yy} = (A_{66}\mathrm{v}^{+} - A_{16}u^{+})/2h\Omega \\ &\sigma_{xx} = -[(a_{16}A_{11} - a_{12}A_{16})u^{+} + (a_{12}A_{66} - a_{16}A_{16})\mathrm{v}^{+}]/(2ha_{11}\Omega) \\ &u = \frac{u^{+}}{2h}(y+h),\ \ \mathrm{v} = \frac{\mathrm{v}^{+}}{2h}(y+h) \end{aligned} \tag{1.12}$$

For the orthotropic beam-strip ($a_{16} = A_{16} = 0$)

$$\sigma_{xy} = G_{12}\frac{u^{+}}{2h},\ \ \sigma_{yy} = \frac{E_2}{1 - \nu_{12}\nu_{21}}\frac{\mathrm{v}^{+}}{2h},\ \ \sigma_{xx} = \frac{E_1\nu_{12}}{1 - \nu_{12}\nu_{21}}\frac{\mathrm{v}^{+}}{2h} \tag{1.13}$$

If $\sigma_{xy}^{+} = \tau^{+} = \mathrm{const},\ \sigma_{yy}^{+} = -\sigma_2^{+} = \mathrm{const}$ from (1.6), (1.11) follows

$$\begin{aligned} &\sigma_{xx} = (a_{12}\sigma_2^{+} - a_{16}\tau^{+})a_{11}^{-1},\ \ \sigma_{xy} = \tau^{+},\ \ \sigma_{yy} = -\sigma_2^{+} \\ &u = (A_{66}\tau^{+} - A_{16}\sigma_2^{+})(y+h), \quad v = (A_{16}\tau^{+} - A_{11}\sigma_2^{+})(y+h) \end{aligned} \tag{1.14}$$

Solution (1.12)-(1.14), as a rule, will not satisfy the boundary conditions at $x = 0, \ell$. The arising residual is removed by the solution for the boundary layer, which is built as in the case of the first boundary value problem. In this way the first established in [13] asymptotics (1.6) permitted to solve a new class of problems for which the hypotheses of classical theory are not applicable. Asymptotics (1.6) is true for layered beams and bars. Having solved equation (1.7) for the arbitrary k -layer and satisfied the contact conditions among the layers, it is not difficult to write out the solution of the inner problem. This asymptotics may be generalized for finding the solutions of the corresponding space problems of anisotropic plates and shells as well, and layered in addition. Let's illustrate what has been said above on the examples of two-layered plates and shell. In case of plates it is required to find the solution of thermoelasticity theory equations in the region $D=\{(\alpha,\beta,\gamma):\alpha,\beta\in\Omega_0,-h_2\le\gamma\le h_1,\ h=\max(h_1,h_2),h<<\ell\}$, ℓ is characteristic tangential dimension of the plate in the plane Ω_0 interface of the layers, when on the facial surface $\gamma=-h_2$ the displacement vector is given and on the opposite surface $\gamma=h_1$ the stress tensor $\sigma^{+}_{j\gamma}$, the displacement vector components or mixed conditions (Fig. 1). Passing in the equations and correlations of elasticity to dimensionless coordinates $\xi=\alpha/\ell,\ \eta=\beta/\ell,\quad \zeta=\gamma/h$ and displacements $u=u_\alpha/\ell,\ \mathrm{v}=u_\beta/\ell,\ w=u_\gamma/\ell$ for *i*-layer ($i=1,2$) we have [5]

$$\frac{1}{A}\frac{\partial\sigma^{(i)}_{\alpha\alpha}}{\partial\xi}+\frac{1}{B}\frac{\partial\sigma^{(i)}_{\alpha\beta}}{\partial\eta}+\varepsilon^{-1}\frac{\partial\sigma^{(i)}_{\alpha\gamma}}{\partial\zeta}+\ell(\sigma^{(i)}_{\alpha\alpha}-\sigma^{(i)}_{\beta\beta})k_\beta+2\ell k_\alpha\sigma^{(i)}_{\alpha\beta}+\ell F^{(i)}_\alpha=0 \qquad (1.15)$$

– –

$$\frac{1}{A}\frac{\partial u^{(i)}}{\partial\xi}+\ell k_\alpha \mathrm{v}^{(i)}=a^{(i)}_{11}\sigma^{(i)}_{\alpha\alpha}+a^{(i)}_{12}\sigma^{(i)}_{\beta\beta}+\cdots+a^{(i)}_{16}\sigma^{(i)}_{\alpha\beta}+\alpha^{(i)}_{11}\theta^{(i)} \qquad (1,2;\alpha,\beta;\xi,\eta;u,\mathrm{v})$$

$$\varepsilon^{-1}\frac{\partial w^{(i)}}{\partial\zeta}=a^{(i)}_{13}\sigma^{(i)}_{\alpha\alpha}+a^{(i)}_{23}\sigma^{(i)}_{\beta\beta}+\cdots+a^{(i)}_{36}\sigma^{(i)}_{\alpha\beta}+\alpha^{(i)}_{33}\theta^{(i)}$$

$$\frac{1}{B}\frac{\partial w^{(i)}}{\partial\eta}+\varepsilon^{-1}\frac{\partial \mathrm{v}^{(i)}}{\partial\zeta}=a^{(i)}_{14}\sigma^{(i)}_{\alpha\alpha}+a^{(i)}_{24}\sigma^{(i)}_{\beta\beta}+\cdots+a^{(i)}_{46}\sigma^{(i)}_{\alpha\beta}+\alpha^{(i)}_{23}\theta^{(i)}$$

where A,B are the coefficients of the first quadratic form, k_α, k_β are the geophysical curvatures, a_{kj} are the constants of elasticity under general anisotropic, α_{kj} are the coefficients of the heat extension, $\theta^{(i)}=T^{(i)}-T^{(i)}_0$ in the temperature differential.

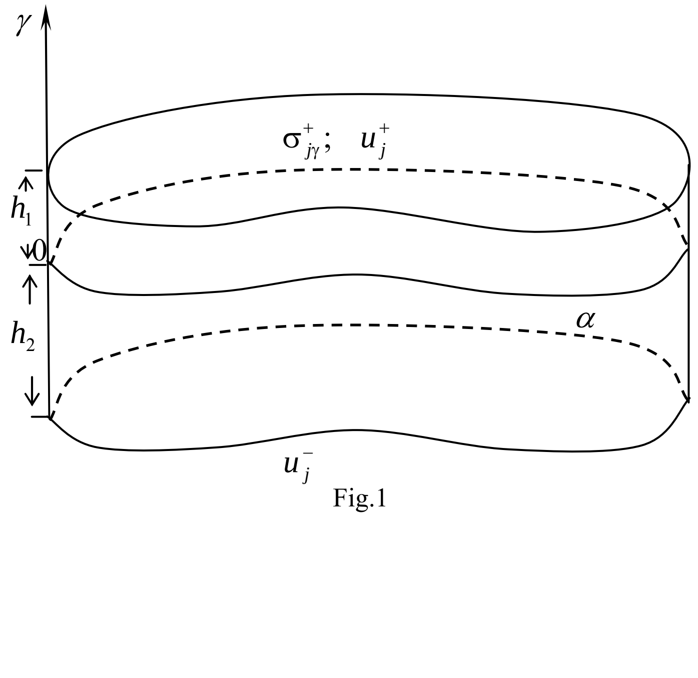

Fig.1

The solution of the inner problem is again sought in the form of (1.6), where $q = -1$ for all the stresses and $q = 0$ for all the displacements. Substituting (1.6) into (1.15) by the above described procedure, iteration process for the sequential determination of all sought values and conditions on the facial surfaces $\gamma = h_1, -h_2$ is obtained. After satisfying these conditions the solution for $\forall s$ becomes well-known.

We reduce this solution for the case, when surface $\gamma = -h_2$ of two-layered plate is rigidly fastened, and on surface $\gamma = h_1$ the load of the constant intensity acts

$$\begin{gathered} u(-h_2) = \mathrm{v}(-h_2) = w(-h_2) = 0 \\ \sigma_{\alpha\gamma}(h_1) = \sigma^+_{\alpha\gamma}, \ \ \sigma_{\beta\gamma}(h_1) = \sigma^+_{\beta\gamma}, \ \ \sigma_{\gamma\gamma}(h_1) = \sigma^+_{\gamma\gamma} \\ \sigma^+_{j\gamma}(h_1) = const, \quad j = \alpha, \beta, \gamma \end{gathered} \tag{1.16}$$

Iteration process breaks on the initial approximation and we have the exact solution

$$\begin{gathered} \sigma^{(i)}_{\alpha\alpha} = A^{(i)}_{13}\sigma^+_{\gamma\gamma} + A^{(i)}_{14}\sigma^+_{\beta\gamma} + A^{(i)}_{15}\sigma^+_{\alpha\gamma} \\ \sigma^{(i)}_{\beta\beta} = A^{(i)}_{23}\sigma^+_{\gamma\gamma} + A^{(i)}_{24}\sigma^+_{\beta\gamma} + A^{(i)}_{25}\sigma^+_{\alpha\gamma} \\ \sigma^{(i)}_{\alpha\beta} = A^{(i)}_{63}\sigma^+_{\gamma\gamma} + A^{(i)}_{64}\sigma^+_{\beta\gamma} + A^{(i)}_{65}\sigma^+_{\alpha\gamma} \\ \sigma^{(i)}_{\alpha\gamma} = \sigma^+_{\alpha\gamma}, \ \ \sigma^{(i)}_{\beta\gamma} = \sigma^+_{\beta\gamma}, \ \ \sigma^{(i)}_{\gamma\gamma} = \sigma^+_{\gamma\gamma} \\ u^{(i)}_{\alpha} = h_2(D^{(i)}_{53}\sigma^+_{\gamma\gamma} + D^{(i)}_{54}\sigma^+_{\beta\gamma} + D^{(i)}_{55}\sigma^+_{\alpha\gamma}) \\ u^{(i)}_{\beta} = h_2(D^{(i)}_{43}\sigma^+_{\gamma\gamma} + D^{(i)}_{44}\sigma^+_{\beta\gamma} + D^{(i)}_{45}\sigma^+_{\alpha\gamma}) \\ u^{(i)}_{\gamma} = h_2(D^{(i)}_{33}\sigma^+_{\gamma\gamma} + D^{(i)}_{34}\sigma^+_{\beta\gamma} + D^{(i)}_{35}\sigma^+_{\alpha\gamma}) \\ D^{(i)}_{kj} = \zeta A^{(i)}_{kj} + A^{(2)}_{kj}, \quad i = 1,2; \ \ \xi = \gamma / h_2 \\ A_{k\ell} = -a_{1\ell}B_{k1} - a_{2\ell}B_{k2} - a_{6\ell}B_{k6}, \ \ \ell, m = 3,4,5 \\ A_{m\ell} = a_{m1}A_{1\ell} + a_{m2}A_{2\ell} + a_{m6}A_{6\ell} + a_{m\ell}, \quad A_{m\ell} \neq A_{\ell m} \\ B_{ij} = (a_{ik}a_{jk} - a_{ij}a_{kk}) / \Delta, \ \ B_{kk} = (a_{ii}a_{jj} - a^2_{ij}) / \Delta \\ i \neq j \neq k \neq i; \ \ i, j, k = 1,2,6, \quad B_{ij} = B_{ji} \\ \Delta = a_{11}a_{22}a_{66} + 2a_{12}a_{26}a_{16} - a_{11}a^2_{26} - a_{22}a^2_{16} - a_{66}a^2_{12} \end{gathered} \tag{1.17}$$

Mathematically exact solution (1.17) permits us to answer a very important question – if the model of Vinker-Fuss bed coefficient for anisotropic and layered foundations is applicable and how to calculate the bed coefficient. For this we write out the connection among the displacements and stresses on the surface of the contact $(\gamma = 0)$ among the layers, putting down index “c” to them:

$$\begin{gathered} u^{(c)}_{\alpha} = h_2(A^{(2)}_{53}\sigma^{(c)}_{\gamma\gamma} + A^{(2)}_{54}\sigma^{(c)}_{\beta\gamma} + A^{(2)}_{55}\sigma^{(c)}_{\alpha\gamma}), \ \ \sigma^{(c)}_{\gamma\gamma} = \sigma^+_{\gamma\gamma} \\ u^{(c)}_{\beta} = h_2(A^{(2)}_{43}\sigma^{(c)}_{\gamma\gamma} + A^{(2)}_{44}\sigma^{(c)}_{\beta\gamma} + A^{(2)}_{45}\sigma^{(c)}_{\alpha\gamma}), \ \ \sigma^{(c)}_{\beta\gamma} = \sigma^+_{\beta\gamma} \\ u^{(c)}_{\gamma} = h_2(A^{(2)}_{33}\sigma^{(c)}_{\gamma\gamma} + A^{(2)}_{34}\sigma^{(c)}_{\beta\gamma} + A^{(2)}_{35}\sigma^{(c)}_{\alpha\gamma}), \ \ \sigma^{(c)}_{\alpha\gamma} = \sigma^+_{\alpha\gamma} \end{gathered} \tag{1.18}$$

If we consider that the normal load $\sigma^+_{\gamma\gamma}$ only acts, we have

$$u_\alpha^{(c)} = h_2 A_{53}^{(2)} \sigma_{\gamma\gamma}^{(c)}, \quad u_\beta^{(c)} = h_2 A_{43}^{(2)} \sigma_{\gamma\gamma}^{(c)}, \quad u_\gamma^{(c)} = h_2 A_{33}^{(2)} \sigma_{\gamma\gamma}^{(c)} \tag{1.19}$$

i.e. under the effect of the normal load on the surface of the contact tangential displacements arise, which means Vinkler-Fuss model breakdown in case of general anisotropy, it will be more tangible with big values $A_{53}^{(2)}, A_{43}^{(2)}$. For orthotropic and isotropic foundations $A_{53}^{(2)} = A_{43}^{(2)} = 0$ and

$$\sigma_{\gamma\gamma}^{(c)} = K u_\gamma^{(c)}, \quad K = \frac{1}{h_2 A_{33}^{(2)}} \tag{1.20}$$

For orthotropic foundation according to (1.17)

$$K = \frac{(1 - \nu_{\alpha\beta}\nu_{\beta\alpha})E_\gamma}{h_2(1 - \nu_{\beta\gamma}\nu_{\gamma\beta} - \nu_{\alpha\gamma}\nu_{\gamma\alpha} - \nu_{\alpha\beta}\nu_{\beta\alpha} - 2\nu_{\alpha\beta}\nu_{\beta\gamma}\nu_{\gamma\alpha})} \tag{1.21}$$

For isotropic foundation

$$K = \frac{(1-\nu)E}{h_2(1+\nu)(1-2\nu)} \tag{1.22}$$

Coefficient K coincides with the well-known bed coefficient, reduced by very difficult way [14].

Considering by asymptotic method multi-layered and inhomogeneous by thickness foundations, we obtain [5]

$$K_n = \frac{1}{\sum_{i=1}^{n} h_i A_{33}^{(i)}}, \quad K = \frac{1}{\int_0^h A_{33}(\gamma) d\gamma} \tag{1.23}$$

where h is the general thickness of the foundation.

Combining Flaman solution for half-plane and Bussinesk solution for half-space with asymptotic solution for strip and layer, it is possible to find the solutions for layered foundations under concentrated and sectionally continuous force effects [15].

Asymptotics (1.6) established for beams and plates is true for shells as well, but the iteration process in simplest cases does not already break on definite approximation, by virtue of which the solution of the inner problem is asymptotic. In general case the solution is fully determined and is expressed through the boundary functions during the satisfaction of the conditions on facial surfaces of the shell [5]. As an illustration we reduce the solution for orthotropic cylindrical shell the outer surface of which is rigidly fastened and inside the constant pressure P (Fig.2).

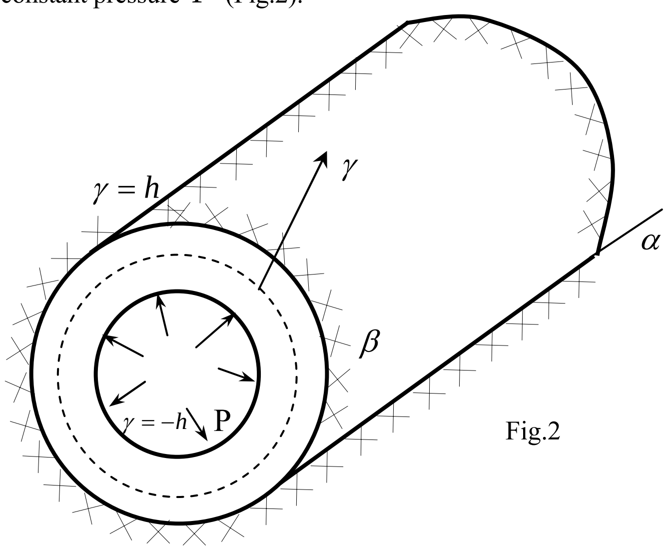

Fig.2

Let ℓ be the length along the generator (element), and α, β be the length of the arch directing the cylinder, then $A = B = 1$, $k_\alpha = k_\beta = 0$, $1/R_1 = 0$, $R_2 = R$, R is the radius of the middle surface of the cylinder. Taking into account that volume forces are lacking, and temperature field does not change and bounded with exactness $O(\varepsilon^2)$ we have solution:

$$
\begin{aligned}
&\sigma_{\alpha\gamma} = \sigma_{\beta\gamma} = \sigma_{\alpha\beta} = 0,\ \ u_\alpha = u_\beta = 0,\\
&\sigma_{\gamma\gamma} = -\frac{R-h}{R+\gamma}P - A_{23}\frac{\gamma+h}{R+\gamma}P,\\
&\sigma_{\alpha\alpha} = -[A_{13}(R-h) + A_{13}A_{23}(h+\gamma) - \frac{a_{12}}{\Delta}A_{33}(\gamma-h)]\frac{P}{R+\gamma}\\
&\sigma_{\beta\beta} = -[A_{23}(R-h) + A_{23}^2(h+\gamma) + \frac{a_{11}}{\Delta}A_{33}(\gamma-h)]\frac{P}{R}\\
&u_\gamma = -A_{33}(\gamma-h)(R-h+hA_{23})\frac{P}{R} + \frac{1}{2}A_{33}(\gamma^2-h^2)\frac{P}{R} - A_{23}A_{33}(\gamma-h)\frac{Ph}{R}\\
&\Delta = a_{11}a_{22} - a_{12}^2
\end{aligned}
\qquad (1.24)
$$

The general asymptotic solution can be written out for layered shells, as well.

III. Asymptotic solution of space dynamic problems for plates and shells.

1. Let's consider the following two interesting types of forced vibrations of orthotropic plates $D = \{(x,y,z): 0 \le x \le a, 0 \le y \le b, -h \le z \le h, \min(a,b) = \ell, \ell >> h\}$:

a) vibrations of plates fastened with absolutely rigid plane foundation:

$$\sigma_{\alpha z}(x,y,h) = \sigma_{\alpha z}^{+}(x,y)\exp(i\Omega t),\quad \alpha = x,y,z \qquad (1.1)$$

$$u(x,y,-h) = 0, \mathrm{v}(x,y,-h) = 0,\ w(x,y,-h) = 0 \qquad (1.2)$$

b) vibrations, caused by the displacement vector, applied to the facial surface of $z = -h$ plate:

$$u(x,y,-h) = u^{-}(x,y)\exp(i\Omega t),\quad (u,\mathrm{v},w) \qquad (1.3)$$

$$u(x,y,h) = 0,\quad (u,\mathrm{v},w) \qquad (1.4)$$

or

$$\sigma_{xz}(x,y,h) = \sigma_{yz}(x,y,h) = \sigma_{zz}(x,y,h) = 0 \qquad (1.5)$$

where $\sigma_{\alpha z}^{+}, u^{-}, \mathrm{v}^{-}, w^{-}$ are the given functions, Ω is the frequency of the forcing effect. Conditions (1.3),(1.4), particularly, simulate seismic effects on the foot of the bases of the constructions, and (1.3), (1.5) - on the flying-landing areas.

It is required to find solutions of dynamic equations of elasticity theory of an orthotropic body:

$$
\begin{aligned}
&\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{xy}}{\partial y} + \frac{\partial\sigma_{xz}}{\partial z} = \rho\frac{\partial^2 u}{\partial t^2},\quad (x,y,z;u,\mathrm{v},w)\\
&\frac{\partial u}{\partial x} = a_{11}\sigma_{xx} + a_{12}\sigma_{yy} + a_{13}\sigma_{zz},\quad (1,2,3;x,y,z;u,\mathrm{v},w)\\
&\frac{\partial u}{\partial y} + \frac{\partial \mathrm{v}}{\partial x} = a_{66}\sigma_{xy},\ \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = a_{55}\sigma_{xz},\ \frac{\partial \mathrm{v}}{\partial z} + \frac{\partial w}{\partial y} = a_{44}\sigma_{yz}
\end{aligned}
\qquad (1.6)
$$

satisfying the corresponding groups of conditions (1.1)-(1.5) and also the conditions on the lateral surface of the plate as well, which are not going to be defined concretely by us, because as in the static problem the appearance of the boundary layers corresponds to them. The solutions of the formulated problem will be sought in the form of

$$\sigma_{\alpha\beta}(x,y,z,t)=\sigma_{jk}(x,y,z)\exp(i\Omega t)$$
$$(u,\mathrm{v},w)=(u_x(x,y),u_y(x,y,z),u_z(x,y,z))\exp(i\Omega t) \tag{1.7}$$
$$\alpha,\beta=x,y,z;\ j,k=1,2,3$$

Substituting (1.7) into (1.6), then passing to dimensionless coordinates $\xi=x/\ell,\eta=y/\ell,\ \zeta=z/\ell$ and displacements $U=u_x/\ell,\ V=u_y/\ell,\ W=u_z/\ell$ we obtain

$$\frac{\partial\sigma_{11}}{\partial\xi}+\frac{\partial\sigma_{12}}{\partial\eta}+\varepsilon^{-1}\frac{\partial\sigma_{13}}{\partial\zeta}+\varepsilon^{-2}\Omega_*^2U=0$$
$$\frac{\partial\sigma_{12}}{\partial\xi}+\frac{\partial\sigma_{22}}{\partial\eta}+\varepsilon^{-1}\frac{\partial\sigma_{23}}{\partial\zeta}+\varepsilon^{-2}\Omega_*^2V=0$$
$$\frac{\partial\sigma_{13}}{\partial\xi}+\frac{\partial\sigma_{23}}{\partial\eta}+\varepsilon^{-1}\frac{\partial\sigma_{33}}{\partial\zeta}+\varepsilon^{-2}\Omega_*^2W=0$$
$$\frac{\partial U}{\partial\xi}=a_{11}\sigma_{11}+a_{12}\sigma_{22}+a_{13}\sigma_{33},\ \ (1,2;\xi,\eta;U,V)$$
$$\varepsilon^{-1}\frac{\partial W}{\partial\zeta}=a_{13}\sigma_{11}+a_{23}\sigma_{22}+a_{33}\sigma_{33} \tag{1.8}$$
$$\frac{\partial U}{\partial\eta}+\frac{\partial V}{\partial\xi}=a_{66}\sigma_{12},\qquad \frac{\partial W}{\partial\xi}+\varepsilon^{-1}\frac{\partial U}{\partial\zeta}=a_{55}\sigma_{13}$$
$$\frac{\partial W}{\partial\eta}+\varepsilon^{-1}\frac{\partial V}{\partial\zeta}=a_{44}\sigma_{23},\ ,\quad \Omega_*^2=\rho h^2\Omega^2,\quad \varepsilon=h/\ell$$

The solution of the singularly perturbed system (1.8) is again combined from the solution of the inner problem (I^{int}) and the boundary layer $(I_b):I=I^{\mathrm{int}}+I_b$. The solution of the inner problem is sought in the form of

$$\sigma_{jk}^{\mathrm{int}}=\varepsilon^{-1+s}\sigma_{jk}^{(s)}(\xi,\eta,\zeta),\ \ j,k=1,2,3;s=\overline{0,N}$$
$$(U^{-\mathrm{int}},V^{\mathrm{int}},W^{\mathrm{int}})=\varepsilon^s(U^{(s)},V^{(s)},W^{(s)}) \tag{1.9}$$

Substituting (1.9) into (1.8), from the new system all the stresses will be expressed through the displacements according to the formulae

$$\sigma_{11}^{(s)}=-A_{23}\frac{\partial W^{(s)}}{\partial\zeta}+A_{22}\frac{\partial U^{(s-1)}}{\partial\xi}-A_{12}\frac{\partial V^{(s-1)}}{\partial\eta}$$
$$\sigma_{22}^{(s)}=-A_{13}\frac{\partial W^{(s)}}{\partial\zeta}-A_{12}\frac{\partial V^{(s-1)}}{\partial\xi}+A_{33}\frac{\partial V^{(s-1)}}{\partial\eta}$$
$$\sigma_{33}^{(s)}=A_{11}\frac{\partial W^{(s)}}{\partial\zeta}-A_{23}\frac{\partial U^{(s-1)}}{\partial\xi}-A_{13}\frac{\partial V^{(s-1)}}{\partial\eta} \tag{1.10}$$

$$\sigma_{12}^{(s)} = \frac{1}{a_{66}}\left(\frac{\partial U^{(s-1)}}{\partial \eta} + \frac{\partial V^{(s-1)}}{\partial \xi}\right), \quad \sigma_{13}^{(s)} = \frac{1}{a_{55}}\left(\frac{\partial U^{(s)}}{\partial \zeta} + \frac{\partial W^{(s-1)}}{\partial \xi}\right)$$

$$\sigma_{23}^{(s)} = \frac{1}{a_{44}}\left(\frac{\partial V^{(s)}}{\partial \zeta} + \frac{\partial W^{(s-1)}}{\partial \eta}\right), \quad Q^{(m)} \equiv 0, \;\; m < 0$$

$$A_{11} = (a_{11}a_{22} - a_{12}^2)/\Delta, \; A_{22} = (a_{22}a_{33} - a_{23}^2)/\Delta$$

$$A_{33} = (a_{11}a_{33} - a_{13}^2)/\Delta, \; A_{12} = (a_{33}a_{12} - a_{13}a_{23})/\Delta$$

$$A_{13} = (a_{11}a_{23} - a_{12}a_{13})/\Delta, \; A_{23} = (a_{22}a_{13} - a_{12}a_{23})/\Delta$$

$$\Delta = a_{11}a_{22}a_{33} + 2a_{12}a_{23}a_{13} - a_{11}a_{23}^2 - a_{22}a_{13}^2 - a_{33}a_{12}^2$$

Substituting the values $\sigma_{13}^{(s)}, \sigma_{23}^{(s)}, \; \sigma_{33}^{(s)}$ into the corresponding first three equations (1.8), we obtain equations to determine the functions $U^{(s)}, V^{(s)}, W^{(s)}$:

$$\frac{\partial^2 U^{(s)}}{\partial \zeta^2} + a_{55}\Omega_*^2 U^{(s)} = R_u^{(s)}, \quad (U, V; a_{55}, a_{44}; R_u, R_v)$$
$$A_{11}\frac{\partial^2 W^{(s)}}{\partial \zeta^2} + \Omega_*^2 W^{(s)} = R_w^{(s)} \tag{1.11}$$

where

$$R_u^{(s)} = -\frac{\partial^2 W^{(s-1)}}{\partial \xi \partial \zeta} - a_{55}\left(\frac{\partial \sigma_{11}^{(s-1)}}{\partial \xi} + \frac{\partial \sigma_{12}^{(s-1)}}{\partial \eta}\right), \quad (u, \mathrm{v}; \xi, \zeta; a_{55}, a_{44}; 1,2)$$
$$R_w^{(s)} = A_{23}\frac{\partial^2 U^{(s-1)}}{\partial \xi \partial \zeta} + A_{13}\frac{\partial^2 V^{(s-1)}}{\partial \eta \partial \zeta} - \left(\frac{\partial \sigma_{13}^{(s-1)}}{\partial \xi} + \frac{\partial \sigma_{23}^{(s-1)}}{\partial \eta}\right), \tag{1.12}$$

The solution of the system (1.11) is

$$U^{(s)} = U_0^{(s)} + u_\tau^{(s)}, \quad (U, V, W) \tag{1.13}$$

where the first is the summand solution of homogeneous equations, and the second is the particular solution of inhomogeneous equations (1.11). Satisfying each group of the boundary conditions (1.1)-(1.5) we have the final solution. The solution

$$U^{(s)} = \frac{1}{\cos 2\Omega_* \sqrt{a_{55}}}[-u_\tau^{(s)}(\xi, \eta, -1)\cos \Omega_* \sqrt{a_{55}}(1 - \zeta) +$$
$$+ \frac{\sqrt{a_{55}}}{\Omega_*}(\sigma_{13}^{+(s)} - \sigma_{13\tau}^{(s)}(\xi, \eta, 1))\sin \Omega_* \sqrt{a_{55}}(1 + \zeta)] + u_\tau^{(s)}(\xi, \eta, \zeta)$$

$$\sigma_{13}^{(s)} = \frac{\Omega_*}{\sqrt{a_{55}}\cos 2\Omega_* \sqrt{a_{55}}}[-u_\tau^{(s)}(\xi, \eta, -1)\sin \Omega_* \sqrt{a_{55}}(1 - \zeta) +$$
$$+ \frac{\sqrt{a_{55}}}{\Omega_*}(\sigma_{13}^{+(s)} - \sigma_{13\tau}^{(s)}(\xi, \eta, 1))\cos \Omega_* \sqrt{a_{55}}(1 + \zeta)] + \sigma_{13\tau}^{(s)}(\xi, \eta, \zeta)$$

$$(U, V; a_{55}, a_{44}; 1,2)$$

$$W^{(s)} = \frac{1}{\cos\frac{2\Omega_*}{\sqrt{A_{11}}}}[-w_\tau^{(s)}(\xi,\eta,-1)\cos\frac{\Omega_*}{\sqrt{A_{11}}}(1-\zeta)+ \tag{1.14}$$

$$+\frac{1}{\Omega_*\sqrt{A_{11}}}(\sigma_{33}^{+(s)}-\sigma_{33\tau}^{(s)}(\xi,\eta,1))\sin\frac{\Omega_*}{\sqrt{A_{11}}}(1+\zeta)]+w_\tau^{(s)}(\xi,\eta,\zeta)$$

$$\sigma_{33}^{(s)} = \frac{\Omega_*\sqrt{A_{11}}}{\cos\frac{2\Omega_*}{\sqrt{A_{11}}}}[-w_\tau^{(s)}(\xi,\eta,-1)\sin\frac{\Omega_*}{\sqrt{A_{11}}}(1-\zeta)+$$

$$+\frac{1}{\Omega_*\sqrt{A_{11}}}(\sigma_{33}^{+(s)}-\sigma_{33\tau}^{(s)}(\xi,\eta,1))\cos\frac{\Omega_*}{\sqrt{A_{11}}}(1+\zeta)]+\sigma_{33\tau}^{(s)}(\xi,\eta,\zeta)$$

$$\sigma_{j3}^{+(0)} = \varepsilon\sigma_{j3}^{+},\quad \sigma_{j3}^{+(s)} = 0,\ s\neq 0,\ j=1,2,3$$

$$\sigma_{13\tau}^{(s)} = \frac{1}{a_{55}}\left[\frac{\partial U_\tau^{(s)}}{\partial\zeta}+\frac{\partial W^{(s-1)}}{\partial\xi}\right],\quad (1,2;\ a_{55},a_{44};\quad \xi,\eta)$$

$$\sigma_{33\tau}^{(s)} = A_{11}\frac{\partial W_\tau^{(s)}}{\partial\zeta}-A_{23}\frac{\partial U^{(s-1)}}{\partial\xi}-A_{13}\frac{\partial V^{(s-1)}}{\partial\eta}$$

corresponds to the conditions (1.1), (1.2).

The solution (1.7), (1.9), (1.14) will be finite, if

$$\cos 2\Omega_*\sqrt{a_{55}}\neq 0,\quad (a_{55},a_{44},1/A_{11}) \tag{1.15}$$

If Ω is so that one of the three conditions is not fulfilled, resonance takes place, such values Ω coincide with the principal values of the frequencies of the free vibrations.

The solution

$$U^{(s)} = \frac{1}{\sin 2\Omega_*\sqrt{a_{55}}}[(u^{-(s)}-u_\tau^{(s)}(\zeta=-1))\sin\Omega_*\sqrt{a_{55}}(1-\zeta)-$$

$$-u_\tau^{(s)}(\zeta=1)\sin\Omega_*\sqrt{a_{55}}(1+\zeta)]+u_\tau^{(s)}(\xi,\eta,\zeta);\quad (U,V;a_{55},a_{44})$$

$$W^{(s)} = \frac{1}{\sin\frac{2\Omega_*}{\sqrt{A_{11}}}}[(w^{-(s)}-w_\tau^{(s)}(\zeta=-1))\sin\frac{\Omega_*}{\sqrt{A_{11}}}(1-\zeta)- \tag{1.16}$$

$$-w_\tau^{(s)}(\zeta=1)\sin\frac{\Omega_*}{\sqrt{A_{11}}}(1+\zeta)]+w_\tau^{(s)}(\xi,\eta,\zeta)$$

$$u^{-(0)} = u^-/\ell,\quad u^{-(s)}=0,\ s\neq 0,\ (u,\mathrm{v},w)$$

corresponds to the conditions (1.3), (1.4).

Under the conditions (1.3), (1.5) we have

$$U^{(s)} = \frac{1}{\cos 2\Omega_* \sqrt{a_{55}}} [(u^{-(s)} - u_\tau^{(s)}(\zeta = -1)) \cos \Omega_* \sqrt{a_{55}} (1-\zeta) -$$

$$- \frac{\sqrt{a_{55}}}{\Omega_*} \sigma_{13\tau}^{(s)}(\zeta = 1) \sin \Omega_* \sqrt{a_{55}} (1+\zeta)] + u_\tau^{(s)}(\xi, \eta, \zeta);$$

$$(U, V; a_{55}, a_{44}; 1, 2) \tag{1.17}$$

$$W^{(s)} = \frac{1}{\cos \frac{2\Omega_*}{\sqrt{A_{11}}}} [(w^{-(s)} - w_\tau^{(s)}(\zeta = -1)) \cos \frac{\Omega_*}{\sqrt{A_{11}}} (1-\zeta) -$$

$$- \frac{1}{\Omega_* \sqrt{A_{11}}} \sigma_{33\tau}^{(s)}(\zeta = 1) \sin \frac{\Omega_*}{\sqrt{A_{11}}} (1+\zeta)] + w_\tau^{(s)}(\xi, \eta, \zeta)$$

The stresses are calculated by formulae (1.7), (1.9), (1.10), and $\sigma_{13\tau}^{(s)}, \sigma_{23\tau}^{(s)}, \sigma_{33\tau}^{(s)}$ -by formulae (1.14). These solutions will be finite, if corresponding to

$$\sin 2\Omega_* \sqrt{a_{55}} \neq 0, \quad \sin 2\Omega_* \sqrt{a_{44}} \neq 0, \quad \sin \frac{2\Omega_*}{\sqrt{A_{11}}} \neq 0 \tag{1.18}$$

$$\cos 2\Omega_* \sqrt{a_{55}} \neq 0, \quad \cos 2\Omega_* \sqrt{a_{44}} \neq 0, \quad \cos \frac{2\Omega_*}{\sqrt{A_{11}}} \neq 0 \tag{1.19}$$

The values Ω under which at least one of the conditions (1.18) or (1.19) is not fulfilled correspond to the resonance. These values coincide with the principal values of the free vibrations frequencies of the plate, under the corresponding homogeneous conditions on the facial planes $z = \pm h$. Note an important fact-if the entering boundary conditions functions $\sigma_{jy}^{+}, u^{-}, \mathrm{v}^{-}, w^{-}$ are polynomials from ξ, η, iteration process breaks, as a result in the inner dynamic problem we obtain an exact solution (solution for the layer).

The above described approach can be used to solve problems on forced vibrations of layered plates. For this, equations (1.6) for an arbitrary layer with number "k" are solved, the structure of solution (1.7), (1.9)-(1.13) remains unchangeable, index "k" is only ascribed to all the values, then the boundary conditions on the facial surfaces and the conditions of full contact between the layers are satisfied. As an illustration we bring the solution for a two-layered orthotropic plate

$D = \{(x, y, z) : 0 \le x \le a, 0 \le y \le b, -h_2 \le z \le h_1, \min(a, b) = \ell, h_1 + h_2 = h, h << \ell\}$,

corresponding conditions (1.3) at $z = -h_2$ and conditions (1.4) at $z = h_1$, when $u^{-}, \mathrm{v}^{-}, w^{-} = const$. The iteration breaks at the initial approach and the following exact solution is obtained:

$$u_k = \tilde{u}_k \exp i\Omega t, \ (u, \mathrm{v}, w), \ \sigma_{jmk} = \tilde{\sigma}_{jmk} \exp i\Omega t, \ k = I, II \tag{1.20}$$

for the first layer ($0 \le \zeta \le \zeta_1, \ \zeta_1 = h_1 / h$)

$$\tilde{u}_I = u^- \sqrt{\frac{\rho_{II}}{a_{55}^{II}}} \frac{\Omega_*}{\Delta_1} \sin a_1^I \Omega_* (\zeta_1 - \zeta)$$

$$\tilde{v}_I = v^- \sqrt{\frac{\rho_{II}}{a_{44}^{II}}} \frac{\Omega_*}{\Delta_2} \sin a_2^I \Omega_* (\zeta_1 - \zeta) \quad (1.21)$$

$$\tilde{w}_I = w^- \sqrt{A_{11}^{II} \rho_{II}} \frac{\Omega_*}{\Delta_3} \sin a_3^I \Omega_* (\zeta_1 - \zeta)$$

$$\tilde{\sigma}_{13I} = \frac{1}{h a_{55}^I} \frac{\partial \tilde{u}_I}{\partial \zeta}, \ \tilde{\sigma}_{23I} = \frac{1}{h a_{44}^I} \frac{\partial \tilde{v}_I}{\partial \zeta}, \ \tilde{\sigma}_{33I} = \frac{A_{11}^I}{h} \frac{\partial \tilde{w}_I}{\partial \zeta}$$

$$\tilde{\sigma}_{12I} = 0, \ \tilde{\sigma}_{11I} = -\frac{A_{23}^I}{h} \frac{\partial \tilde{w}_I}{\partial \zeta}, \ \tilde{\sigma}_{22I} = -\frac{A_{13}^I}{h} \frac{\partial \tilde{w}_I}{\partial \zeta}$$

for the second layer ($-\zeta_2 \le \zeta \le 0, \ \zeta_2 = h_2 / h$)

$$\tilde{u}_{II} = \left[\sqrt{\frac{\rho_{II}}{a_{55}^{II}}} \sin a_1^I \Omega_* \zeta_1 \cos a_1^{II} \Omega_* \zeta - \sqrt{\frac{\rho_I}{a_{55}^I}} \cos a_1^I \Omega_* \zeta_1 \sin a_1^{II} \Omega_* \zeta \right] \frac{\Omega_*}{\Delta_1} u^-$$

$$\tilde{v}_{II} = \left[\sqrt{\frac{\rho_{II}}{a_{44}^{II}}} \sin a_2^I \Omega_* \zeta_1 \cos a_2^{II} \Omega_* \zeta - \sqrt{\frac{\rho_I}{a_{44}^I}} \cos a_2^I \Omega_* \zeta_1 \sin a_2^{II} \Omega_* \zeta \right] \frac{\Omega_*}{\Delta_2} v^-$$

$$\tilde{w}_{II} = \left[\sqrt{A_{11}^{II} \rho_{II}} \sin a_3^I \Omega_* \zeta_1 \cos a_3^{II} \Omega_* \zeta - \sqrt{A_{11}^I \rho_I} \cos a_3^I \Omega_* \zeta_1 \sin a_3^{II} \Omega_* \zeta \right] \frac{\Omega_*}{\Delta_3} w^-$$

$$\tilde{\sigma}_{13II} = \frac{1}{h a_{55}^{II}} \frac{\partial \tilde{u}_{II}}{\partial \zeta}, \ \tilde{\sigma}_{23II} = \frac{1}{h a_{44}^{II}} \frac{\partial \tilde{v}_{II}}{\partial \zeta}, \ \tilde{\sigma}_{33II} = \frac{A_{11}^{II}}{h} \frac{\partial \tilde{w}_{II}}{\partial \zeta}$$

$$\tilde{\sigma}_{12II} = 0, \ \tilde{\sigma}_{11II} = -\frac{A_{23}^{II}}{h} \frac{\partial \tilde{w}_{II}}{\partial \zeta}, \ \tilde{\sigma}_{22II} = -\frac{A_{13}^{II}}{h} \frac{\partial \tilde{w}_{II}}{\partial \zeta} \quad (1.22)$$

$$a_1^k = \sqrt{a_{55}^k \rho_k}, \ a_2^k = \sqrt{a_{44}^k \rho_k}, \ a_3^k = \sqrt{\frac{\rho_k}{A_{11}^k}}, \ k = I, II$$

$$\Delta_1 = \left(\sqrt{\frac{\rho_I}{a_{55}^I}} \cos a_1^I \Omega_* \zeta_1 \sin a_1^{II} \Omega_* \zeta_2 + \sqrt{\frac{\rho_{II}}{a_{55}^{II}}} \sin a_1^I \Omega_* \zeta_1 \cos a_1^{II} \Omega_* \zeta_2 \right) \Omega_*$$

$$\Delta_2 = \left(\sqrt{\frac{\rho_I}{a_{44}^I}} \cos a_2^I \Omega_* \zeta_1 \sin a_2^{II} \Omega_* \zeta_2 + \sqrt{\frac{\rho_{II}}{a_{44}^{II}}} \sin a_2^I \Omega_* \zeta_1 \cos a_2^{II} \Omega_* \zeta_2 \right) \Omega_*$$

$$\Delta_3 = \left(\sqrt{A_{11}^I \rho_I} \cos a_3^I \Omega_* \zeta_1 \sin a_3^{II} \Omega_* \zeta_2 + \sqrt{A_{11}^{II} \rho_{II}} \sin a_3^I \Omega_* \zeta_1 \cos a_3^{II} \Omega_* \zeta_2 \right) \Omega_*$$

Solution (1.20)-(1.22) will be finite, if

$$\Delta_1 \ne 0, \ \Delta_2 \ne 0, \ \Delta_3 \ne 0 \quad (1.23)$$

otherwise a resonance will arise and the corresponding values Ω will coincide with the main values of the frequencies of two-layered plate free vibrations.

This result can have an obvious application in seismic steady construction. The two-layered plate simulates the packet base-foundation of the constructions. The power (thickness) and the elastic characteristics of the compressed layer (foundation) are usually known, the region of frequencies Ω change of the outer dynamic (seismic) effect is known, as well. Using formulae (1.22) for $\Delta_1, \Delta_2, \Delta_3$ the parameters of the first layer (base) can be chosen so, that conditions (1.23) were fulfilled, i.e. the construction from the very beginning wouldn't fall into resonance condition.

By the described above procedure the solutions for three-layered and multilayered plates are written out. The analysis of the asymptotic solution of the corresponding three-dimensional boundary problems for three-layered plates reduces to the important conclusions too.

It is established, that when the displacement vector, which changes harmonically in time, is applied to the facial surface of the lower-third layer, and the upper layer is rigidly fastened or free, then if all the three layers consist of rigid similar materials, the amplitudes of the vibrations grow, though negligibly from layer to later. And in the presence of the middle layer from softer material (for example, rubber) the vibrations amplitudes in the upper layer, particularly tangential vibrations, diminish abruptly [16]. The established fact proves the necessity of application of seismoisolators in the seismic construction, as when building the constructions, if between the concrete base and foundation a thin layer of rubber like soft material is inserted, it will bring to diminution of dangerous vibrations in the base during the earthquakes and, as consequence and to the increase of the construction seismosteadiness.

In case of the first dynamic boundary value problem of elasticity theory for strips-beams, plates and shells (on the facial surfaces the corresponding stresses tensor components are given) the asymptotics of the static problem: $q_{\sigma_{xx}} = q_{\sigma_{xy}} = q_{\sigma_{yy}} = -2,\ q_{\sigma_{xz}} = q_{\sigma_{yz}} = -1,\ q_{\sigma_{zz}} = 0,$ $q_u = q_{\mathrm{v}} = -2,\ q_w = -3$ reduces to contradictory correlations, but the above brought asymptotics (1.7), (1.9) of mixed problem [17] passes. In this sense the asymptotics (1.7), (1.9) is universal. The procedure of finding the general solution is unchangeable [17].

IV Asymptotics of free vibrations of plates and shells.

1. In the previous chapter we noted the particular role of free vibrations frequencies for arise of resonance states. The asymptotic method permits to determine the values of frequencies and forms of the free vibrations of known and anisotropic plates and shells on the base of the three-dimensional problem dynamic equations of elasticity theory. Consider variants of the free vibrations of plates and shells representing the greatest interest. Set the problem: to find the frequencies of the free vibrations and the free functions of an orthotropic plate $D = \{(x, y, z):$ $0 \le x \le a,\ 0 \le y \le b, -h \le z \le h,\ \min(a,b) = \ell, l >> h\}$ corresponding to the following conditions of the facial surfaces

$$u(x, y, -h) = \mathrm{v}(x, y, -h) = w(x, y, -h) = 0 \tag{1.1}$$

$$u(x, y, h) = \mathrm{v}(x, y, h) = w(x, y, h) = 0 \tag{1.2}$$

or

$$\sigma_{xz}(x, y, h) = \sigma_{yz}(x, y, h) = \sigma_{zz}(x, y, h) = 0 \tag{1.3}$$

For this it is necessary to find nontrivial solutions of dynamic equations (III.1.6) under the conditions (1.1), (1.2) or (1.1), (1.3). Below we shall be convinced, that the conditions on the lateral surface don't influence on the values of the free vibrations frequencies, the vibrations in the boundary layer with the same frequency of free vibrations in the inner problem correspond them. The solutions of the set problem will be sought in the form of

$$\sigma_{\alpha\beta}(x,y,z,t)=\sigma_{jk}(x,y,z)\exp(i\omega t)$$
$$(u,\mathrm{v},w)=(u_x(x,y,z),u_y(x,y,z),u_z(x,y,z))\exp(i\omega t) \tag{1.4}$$
$$\alpha,\beta=x,y,z;\ \ j,k=1,2,3$$

where ω is the frequency of the free vibrations. When passing to dimensionless coordinates $\xi=x/\ell,\ \eta=y/\ell,\ \zeta=z/h$ and displacements $U=u_x/\ell, V=u_y/\ell,\ W=u_z/\ell$ the dynamic equations of the three-dimensional problem will have the form

$$\frac{\partial\sigma_{11}}{\partial\xi}+\frac{\partial\sigma_{12}}{\partial\eta}+\varepsilon^{-1}\frac{\partial\sigma_{13}}{\partial\zeta}+\varepsilon^{-2}\omega_*^2U=0$$
$$\frac{\partial\sigma_{12}}{\partial\xi}+\frac{\partial\sigma_{22}}{\partial\eta}+\varepsilon^{-1}\frac{\partial\sigma_{23}}{\partial\zeta}+\varepsilon^{-2}\omega_*^2V=0$$
$$\frac{\partial\sigma_{13}}{\partial\xi}+\frac{\partial\sigma_{23}}{\partial\eta}+\varepsilon^{-1}\frac{\partial\sigma_{33}}{\partial\zeta}+\varepsilon^{-2}\omega_*^2W=0$$
$$\frac{\partial U}{\partial\xi}=a_{11}\sigma_{11}+a_{12}\sigma_{22}+a_{13}\sigma_{33},\ \ (1,2;\xi,\eta;U,V) \tag{1.5}$$
$$\varepsilon^{-1}\frac{\partial W}{\partial\zeta}=a_{13}\sigma_{11}+a_{23}\sigma_{22}+a_{33}\sigma_{33}$$
$$\frac{\partial U}{\partial\eta}+\frac{\partial V}{\partial\xi}=a_{66}\sigma_{12},\qquad \frac{\partial W}{\partial\xi}+\varepsilon^{-1}\frac{\partial U}{\partial\zeta}=a_{55}\sigma_{13}$$
$$\frac{\partial W}{\partial\eta}+\varepsilon^{-1}\frac{\partial V}{\partial\zeta}=a_{44}\sigma_{23},\ ,\quad \omega_*^2=\rho h^2\omega^2,\quad \varepsilon=h/\ell$$

The solution of singularly perturbed system (1.5) has the form of $I=I^{\text{int}}+I_b$.

The solution of the inner problem I^{int} will be sought in the form of [18]

$$\sigma_{jk}^{\text{int}}=\varepsilon^{-1+s}\sigma_{jk}^{(s)}(\xi,\eta,\zeta)$$
$$(U^{\text{int}},V^{\text{int}},W^{\text{int}})=\varepsilon^s(U^{(s)},V^{(s)},W^{(s)}) \tag{1.6}$$
$$\omega_*^2=\varepsilon^s\omega_{*s}^2,\ \ s=\overline{0,N}$$

Substituting (1.6) into (1.5) and equalizing the coefficients under the similar degrees ε, we obtain a system form where the stresses will be expressed through the displacements by formulae (III.1.10), the last ones are determined from the equations

$$\frac{\partial^2U^{(s)}}{\partial\zeta^2}+a_{55}\omega_{*k}^2U^{(s-k)}=R_u^{(s)},\quad (U,V;a_{55},a_{44}),\ \ k=\overline{0,s}$$
$$R_u^{(s)}=-\frac{\partial^2W^{(s-1)}}{\partial\xi\partial\zeta}-a_{55}\left[\frac{\partial\sigma_{11}^{(s-1)}}{\partial\xi}+\frac{\partial\sigma_{12}^{(s-1)}}{\partial\eta}\right],\ \ (U,V;\xi,\eta;a_{55},a_{44};1,2) \tag{1.7}$$

$$A_{11}\frac{\partial^2 W^{(s)}}{\partial\zeta^2}+\omega_{*k}^2 W^{(s-k)}=R_w^{(s)},\quad k=\overline{0,s}$$

$$R_w^{(s)}=A_{23}\frac{\partial^2 U^{(s-1)}}{\partial\xi\partial\zeta}+A_{13}\frac{\partial^2 V^{(s-1)}}{\partial\eta\partial\zeta}-\left[\frac{\partial\sigma_{13}^{(s-1)}}{\partial\xi}+\frac{\partial\sigma_{23}^{(s-1)}}{\partial\eta}\right] \tag{1.8}$$

At $s=0,\ R_u^{(0)}=R_v^{(0)}=R_w^{(0)}=0$, equations (1.7), (1.8) become independent. Having solved equation (1.7) and satisfied conditions (1.1), (1.2) for U we obtain a system of homogeneous algebraic equations for the existence of the nonzero solution of which it is necessary to fulfill the equations

$$\sin 2\sqrt{a_{55}}\,\omega_{*0}=0$$

where
$$\omega_{*0n}=\frac{\pi n}{2\sqrt{a_{55}}},\quad n\in N \tag{1.9}$$

or

$$\omega_{0n}^{xz}=\frac{\pi n}{2h\sqrt{\rho a_{55}}}=\frac{\pi n}{2h}\sqrt{\frac{G_{13}}{\rho}},\quad n\in N \tag{1.10}$$

Having satisfied the rest of the conditions (1.1), (1.2), we obtain dispersion equations

$$\sin 2\sqrt{a_{44}}\,\omega_{*0}=0,\quad \sin\frac{2}{\sqrt{A_{11}}}\omega_{*0}=0 \tag{1.11}$$

which correspond to the following values of frequencies

$$\omega_{0n}^{yz}=\frac{\pi n}{2h\sqrt{\rho a_{44}}}=\frac{\pi n}{2h}\sqrt{\frac{G_{23}}{\rho}},\quad n\in N \tag{1.12}$$

$$\omega_{0n}^{p}=\frac{\pi n}{2h}\sqrt{\frac{A_{11}}{\rho}}=\frac{\pi n}{2h}\sqrt{\frac{E_3}{\rho}\frac{1-\nu_{12}\nu_{21}}{1-\nu_{12}\nu_{21}-\nu_{31}(\nu_{12}\nu_{23}+\nu_{13})-\nu_{32}(\nu_{21}\nu_{13}+\nu_{23})}} \tag{1.13}$$

where G_{13},G_{23} are the shear module, $\sqrt{\frac{G_{13}}{\rho}}=V_s^{xz}$, $\sqrt{\frac{G_{23}}{\rho}}=V_s^{yz}$ are the well-known in seismology velocities of shear waves propagations, and $V_p=\sqrt{\frac{A_{11}}{\rho}}$ is the velocity of longitudinal waves. The frequencies (1.10) correspond to the following free functions and solutions:

$$U_{nI}^{(0)}=C_{un}^{(0)}(\xi,\eta)\sin\pi n\zeta \text{ or } U_{nI}^{(0)}=C_{un}^{(0)}(\xi,\eta)\cos\frac{\pi}{2}(2n+1)\zeta \tag{1.14}$$

$$\sigma_{13I}^{(0)}=\frac{1}{a_{55}}\frac{\partial U_{nI}^{(0)}}{\partial\zeta},\ \sigma_{23I}^{(0)}=0,V_{nI}^{(0)}=0,\ \sigma_{11I}^{(0)}=\sigma_{22I}^{(0)}=\sigma_{33I}^{(0)}=\sigma_{12I}^{(0)}=0,W_{nI}^{(0)}=0$$

The frequencies (1.12) correspond to

$$V_{nII}^{(0)}=C_{vn}^{(0)}(\xi,\eta)\sin\pi n\zeta \text{ or } V_{nII}^{(0)}=C_{vn}^{(0)}(\xi,\eta)\cos\frac{\pi}{2}(2n+1)\zeta \tag{1.15}$$

$$\sigma_{23II}^{(0)}=\frac{1}{a_{44}}\frac{\partial V_{nII}^{(0)}}{\partial\zeta},\ \sigma_{13II}^{(0)}=0,\ U_{nII}^{(0)}=0,\ \sigma_{12II}^{(0)}=0,W_{nII}^{(0)}=0,\ \sigma_{11II}^{(0)}=\sigma_{22II}^{(0)}=0$$

In case of the frequencies (1.13) we have

$$W_{nIII}^{(0)} = C_{wn}^{(0)}(\xi,\eta)\sin\pi n\zeta \text{ or } W_{nIII}^{(0)} = C_{wn}^{(0)}(\xi,\eta)\cos\frac{\pi}{2}(2n+1)\zeta \quad (1.16)$$

$$\sigma_{33III}^{(0)} = A_{11}\frac{\partial W_{nIII}^{(0)}}{\partial\zeta},\ \sigma_{11III}^{(0)} = -A_{23}\frac{\partial W_{nIII}^{(0)}}{\partial\zeta},\ \sigma_{22III}^{(0)} = -A_{13}\frac{\partial W_{nIII}^{(0)}}{\partial\zeta}$$

$$U_{nIII}^{(0)} = 0,\ \ \sigma_{13III}^{(0)} = 0,\ V_{nIII}^{(0)} = 0,\ \sigma_{23III}^{(0)} = \sigma_{12III}^{(0)} = 0$$

The free functions $\{\varphi_n\} = \{\sin\pi n\zeta\}$, $\{\psi_n\} = \{\cos\frac{\pi}{2}(2n+1)\zeta\}$ compose an orthonormalized system on the interval [-1;1]. Thus, in the plate two types of vibrations – shear ((1.14), (1.15)) and longitudinal ((1.16)) vibrations arise.

At $s \geq 1$ the solution of the equations (1.10), (1.12) may be sought in the form of series along the free functions $\{\varphi_n\}$ or $\{\psi_n\}$ of the initial approach for each variant (1.10), (1.12), (1.13) of the frequencies values. The calculations show, that

$$u_{nI}^{(1)} = 0,\ \ \omega_{*1nI} = 0,\ \ V_{nI}^{(1)} \neq 0,\ \ W_{nI}^{(1)} \neq 0$$
$$u_{nI}^{(2)} \neq 0,\ \ \omega_{*2nI} \neq 0,\ \ V_{nI}^{(2)} \neq 0,\ \ W_{nI}^{(2)} \neq 0 \quad (1.17)$$

If we are restricted by the approaches $s = 0,1,2$ we have

$$U_{nI} = U_{nI}^{(0)} + \varepsilon^2 U_{nI}^{(2)},\ \ \omega_{*nI}^2 = \omega_{*0nI}^2 + \varepsilon^2\omega_{*2nI}^2$$
$$V_{nI} = \varepsilon V_{nI}^{(1)} + \varepsilon^2 V_{nI}^{(2)},\ \ W_{nI} = \varepsilon W_{nI}^{(1)} + \varepsilon^2 W_{nI}^{(2)} \quad (1.18)$$

From (1.18) two important properties: 1) correction to the basic solution and frequency of the order $O(\varepsilon^2)$, that is why at small ε we may be restricted by the initial approach, 2) shear vibrations of one type generate the shear vibrations of another type and longitudinal vibrations as well and both of them are with the same basic frequency, but the amplitudes accompanying the vibrations are one order smaller than the basic ones. The analogous picture takes place for the frequencies (1.12), (1.13).

Consider the boundary conditions (1.1), (1.3). They correspond to the equations of the frequencies

$$\cos 2\sqrt{a_{55}}\,\omega_{*0} = 0,\quad (a_{55}, a_{44}, \frac{1}{A_{11}}) \quad (1.19)$$

which correspond to the frequencies

$$\omega_{0n}^{I} = \frac{p}{4h\sqrt{\rho a_{55}}}(2n+1) = \frac{\pi}{4h}\sqrt{\frac{G_{13}}{\rho}}(2n+1),\ \ n \in N$$
$$(I, II, III; a_{55}, a_{44}, 1/A_{11}; G_{13}, G_{23}, A_{11}) \quad (1.20)$$

It is not difficult to write out free functions too.

The consideration of free vibrations in the zone of the boundary layer reduces to the conduction, that each frequency, determined from the solution of the inner problem, corresponds to the class of boundary functions, which when removing from the lateral surface into the inside the plate diminish exponentially, i.e. in the zone of the boundary layer under the free vibrations a mixed picture is originated.

2. In case of layered plates the basic equations and correlations of elasticity are the same, the number of layer "k" is only ascribed to all the values. The solution for an arbitrary layer "k" which by its form coincides with the solution of the inner problem for a one-layered plate [19], then the conditions (1.1), (1.2), or (1.1), (1.3) on the facial surfaces of a layered packet and the conditions of full contact among all the layers are satisfied. As a result we obtain a system from homogeneous algebraic equations. The existence condition

of nonzero solution of this system (the determinant equals zero) is the very equation of frequencies of free vibrations. For two-layered orthotropic plates $D=\{(x,y,z);0\le x\le a,0\le y\le b,-h_2\le z\le h_1,\min(a,b)=\ell, \quad h=h_1+h_2, h<<\ell\}$ three variants of independent equations at $s=0$ are obtained

$$\sqrt{\frac{a_{55}^{II}\rho_I}{a_{55}^{I}\rho_{II}}}\sin\sqrt{a_{55}^{I}\rho_I}\,\omega_{*0}\zeta_1\sin\sqrt{a_{55}^{II}\rho_{II}}\,\omega_{*0}\zeta_2-\cos\sqrt{a_{55}^{I}\rho_I}\,\omega_{*0}\zeta_1\cos\sqrt{a_{55}^{II}\rho_{II}}\,\omega_{*0}\zeta_2=0$$

$$(a_{55}^k,a_{44}^k,\frac{1}{A_{11}^k}),\ \zeta_1=h_1/h,\ \zeta_2=h_2/h;\ k=I,II \tag{2.1}$$

Corresponding to the boundary conditions

$$\sigma_{xz}^I=\sigma_{yz}^I=\sigma_{zz}^I=0 \quad \text{for } z=h_1 \tag{2.2}$$

$$u^{II}=\mathrm{v}^{II}=w^{II}=0 \quad \text{for } z=-h_2 \tag{2.3}$$

$$U^{(I,0)}=C_u^{(I,0)}\cos\sqrt{a_{55}^I\rho_I}\,\omega_{*0}(\zeta-\zeta_1)$$

$$U^{(II,0)}=C_u^{(II,0)}\sin\sqrt{a_{55}^{II}\rho_{II}}\,\omega_{*0}(\zeta+\zeta_2) \tag{2.4}$$

$$(a_{55}^k,a_{44}^k,1/A_{11}^k)$$

are free functions.

The equations (2.1) are reduced to the standard form

$$\cos p\omega_{*0}+r\cos q\omega_{*0}=0 \tag{2.5}$$

where

$$p=\zeta_1\sqrt{a_{55}^I\rho_I}+\zeta_2\sqrt{a_{55}^{II}\rho_{II}},\quad q=\zeta_1\sqrt{a_{55}^I\rho_I}-\zeta_2\sqrt{a_{55}^{II}\rho_{II}}$$

$$r=\frac{\sqrt{a_{55}^I\rho_{II}}-\sqrt{a_{55}^{II}\rho_I}}{\sqrt{a_{55}^I\rho_{II}}+\sqrt{a_{55}^{II}\rho_I}},\quad (a_{55}^k,a_{44}^k,1/A_{11}^k),\quad k=I,II \tag{2.6}$$

The roots of which are easily found, if elastic and geometrical parameters of a layered packet are given. Dispersion equations for three-layered and multilayered plates, too, can be obtained.

IF we introduce function φ_m:

$$\varphi_m=\begin{cases}\sqrt{\rho_I}U_m^{(I,0)}, & 0\le\zeta\le\zeta_1\\ \sqrt{\rho_{II}}U_m^{(II,0)}, & -\zeta_2\le\zeta\le 0\end{cases} \tag{2.7}$$

It is proved, that the function $\{\varphi_m\}$ are orthogonal on the interval $[-\zeta_2,\zeta_1]$, i.e. the free functions are orthogonal with the weight.

The described approach of frequencies determination and forms of free vibrations are spread on the shell too [20], particularly, it is established that $\omega_{*1n}\neq 0$, i.e. if restricted by initial approach, the error will be about $O(\varepsilon)$.

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Institute of Mechanics of NAS Armenia
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