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ИЗВЕСТИЯ НАЦИОНАЛЬНОЙ АКАДЕМИИ НАУК АРМЕНИИ

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DUAL INTEGRAL EQUATION SYSTEMS INVOLVING BESSEL FUNCTIONS WITCH ARE ARISING IN ELASTICITY THEORY

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Возникшие в теории упругости системы парных интегральных уравнений, содержащие функции Бесселя

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Առաձգական տեսության մեջ առաջացող Բեսելի ֆունկցիաներ պարունակող զույգ ինտեգրալ հավասարումների համակարգեր

Դիտարկվում է առաձգականության տեսության, ջերմահաղորդականության տեսության, էլեկտրոստատիկայի և պոտենցիալի տեսության մեջ, ինչպես նաև մաթեմատիկական ֆիզիկայի բազմաթիվ այլ բնագավառներում հաճախ հանդիպող տարածական շերտի տեսքով տիրույթում Լապլասի հավասարման համար առանցքահամաչափ խառը եզրային խնդիր, երբ շերտի մի եզրի վրա տրված է որոնելի ֆունկցիայի զրոյական արժեքը, իսկ մյուս եզրագծի կամայական վերջավոր թվով միջակայքերից յուրաքանչյուրի վրա տրված է որոնելի ֆունկցիան ցանկացած ֆունկցիայի տեսքով, իսկ այդ եզրագծի մնացած մասի վրա որոնելի ֆունկցիայի նորմալ ածանցյալի արժեքը ընդունված է հավասար զրոյի: Որոնելի հարմոնիկ ֆունկցիան ներկայացված է Բեսելի առաջին սեռի զրոյական ինդեքսի ֆունկցիան պարունակող Հանկելի ինտեգրալով, որի օգնությամբ նկարագրված խառը եզրային խնդրի լուծումը բերված է Բեսելի նշված ֆունկցիայով զույգ ինտեգրալ հավասարումների համակարգի: Վերջինս իր հերթին Չեբիշևի բազմանդամների օգնությամբ բերվում է պարզ կառուցվածքի գծային հանրահաշվական հավասարումների անվերջ համակարգի:

Dual integral equations method is used for the solution of a wide class of problems in mathematical physics. This method may be considered as a generalization of a separation of variable method for the boundary-value problems with mixed boundary conditions. There are a lot of papers connected with the application of dual integral equations method for the solution of mixed boundary-value problems of potential theory, theory of elasticity, heat conduction theory and electrostatics. The monographs [1,2] include the systematic account of this method and the information about its connection with mixed boundary-value problems of potential theory. A lot of application to mixed boundary-value problems with one boundary line of boundary conditions of the first and second kind are analyzed in this monograph. Further generalization of dual integral equations engendered, in particular, by Hankel and Fourier transforms, is associated with the solution of so called triple integral equations in the case of three intervals with different kinds of boundary conditions [3,4]. Electrostatic problem for a plane circular ring is a typical example of such a kind. The solutions of mixed boundary-value problems are considerably complicated in the case of more than three intervals with different kinds of boundary conditions and it is necessary to elaborate the efficient methods for solution of corresponding integral equations.

In this paper the method for reduction of dual integral equation systems with Hankel kernels to infinite system of algebraic equations is proposed. This method may be used for the solution of various mixed boundary-value problems of axisymmetric potential theory in the case of an arbitrary number of intervals with boundary conditions of the first and second kind.

1. Let us consider an axisymmetric solution of the Laplace equation $u(r,z)$ in the

cylindrical coordinates system in domain of an infinite layer $r \geq 0,\ 0 < z < h$. This solution is satisfied the following condition

$$u(r,0) = 0,\ \ r \geq 0 \tag{1.1}$$

in the plane $z = 0$ and mixed boundary conditions

$$u(r,h) = f_i(r),\ \ a_i < r < b_i,\ \ i = 1,2,...N \tag{1.2}$$

$$\left.\frac{\partial u(r,z)}{\partial z}\right|_{z=h} = 0,\ 0 < r < a_1,\ b_i < r < a_{i+1},\ \ i = 1,2,...N-1;\ \ r > b_N \tag{1.3}$$

in the plane $z = h$.

We suppose the function $u(r,z)$ to be limited for $r \to 0$ and $r \to \infty$, and represent the function $u(r,z)$ in the form of Hankel integral transform

$$u(r,z) = \int_0^\infty C(\xi)\mathrm{sh}\xi z J_0(\xi r)d\xi \tag{1.4}$$

with account of condition (1.1).

Substitution of (1.4) in the mixed boundary conditions (1.2), (1.3) gives the following integral relationships for the determination of the function $C(\xi)$

$$\begin{cases}\int_0^\infty \mathrm{sh}\xi h C(\xi)J_0(\xi r)d\xi = f_i(r),\ \ a_i < r < b_i,\ \ \ i = 1,2,...,N \\ \int_0^\infty \xi\mathrm{ch}\xi h C(\xi)J_0(\xi r)d\xi = 0,\ \ 0 < r < a_1,\ b_i < r < a_{i+1}\ \ i = 1,2,...,N-1,\ r > b_N\end{cases} \tag{1.5}$$

Eqs. (1.5) may be written in the form

$$\int_0^\infty (1 - G(\xi))A(\xi)J_0(\xi r)d\xi = f_i(r),\ \ a_i < r < b_i,\ \ \ i = 1,2,...,N \tag{1.6}$$

$$\int_0^\infty \xi A(\xi)J_0(\xi r)d\xi = 0,\ \ 0 < r < a_1,\ b_i < r < a_{i+1}\ \ \ i = 1,2,...,N-1,\ r > b_N \tag{1.7}$$

where is denoted

$$A(\xi) = C(\xi)\mathrm{ch}\xi h,\ \ \ \ G(\xi) = [\exp(-\xi h)]/[\mathrm{ch}\xi h] \tag{1.8}$$

Passing to solution of the integral Eqs. (1.6), (1.7) we assume

$$\int_0^\infty \xi A(\xi)J_0(\xi r)d\xi = \sum_{n=0}^\infty c_n^{(i)} \frac{\cos n\varphi_i}{\sin\varphi_i},\ \ a_i < r < b_i \tag{1.9}$$

where $c_n^{(i)}\ (i = 1,2,...,N)$ are unknown coefficients and

$$\cos\varphi_i = \frac{\left(\frac{b_i + a_i}{2}\right)^2 + \left(\frac{b_i - a_i}{2}\right)^2 - r^2}{2\left(\frac{b_i + a_i}{2}\right)\left(\frac{b_i - a_i}{2}\right)} = \frac{b_i^2 + a_i^2 - 2r^2}{b_i^2 - a_i^2} \tag{1.10}$$

$$\sin\varphi_i = \frac{2}{(b_i^2 - a_i^2)}\sqrt{(b_i^2 - r^2)(r^2 - a_i^2)} \tag{1.11}$$

and besides

$$r = \sqrt{\left(\frac{b_i + a_i}{2}\right)^2 + \left(\frac{b_i - a_i}{2}\right)^2 - 2\left(\frac{b_i + a_i}{2}\right)\left(\frac{b_i - a_i}{2}\right)\cos\varphi_i} \tag{1.12}$$

for $0 < \varphi_i < \pi$.

Then on the base of (1.7), (1.9) we get the following expression for the function $A(\xi)$

$$A(\xi) = \sum_{i=1}^{N}\sum_{n=0}^{\infty} c_n^{(i)} \int_{a_i}^{b_i} J_0(\xi r)\frac{\cos n\varphi_i}{\sin\varphi_i} r dr \tag{1.13}$$

using the inversion formula of Hankel transform.

In accordance with (1.12) we perform change of integration variable in the right part of (1.13) using the formula

$$rdr = \frac{b_i^2 - a_i^2}{4}\sin\varphi_i d\varphi_i \quad (0 < \varphi_i < \pi)$$

Then (1.13) takes the form

$$A(\xi) = \sum_{i=1}^{N}\left(\frac{b_i^2 - a_i^2}{4}\right)\sum_{n=0}^{\infty} c_n^{(i)} \int_{0}^{\pi} J_0(\xi r)\cos n\varphi_i d\varphi_i \tag{1.14}$$

Using the addition formula for the Bessel function

$$J_0(\xi r) = J_0\left(\xi\frac{b_i + a_i}{2}\right)J_0\left(\xi\frac{b_i - a_i}{2}\right) + 2\sum_{k=1}^{\infty} J_k\left(\xi\frac{b_i + a_i}{2}\right)J_k\left(\xi\frac{b_i - a_i}{2}\right)\cos k\varphi_i$$

$$(a_i < r < b_i) \tag{1.15}$$

we obtain

$$A(\xi) = (\pi/4)\sum_{i=1}^{N}\left(b_i^2 - a_i^2\right)\sum_{n=0}^{\infty} c_n^{(i)} J_n\left(\xi\frac{b_i + a_i}{2}\right)J_n\left(\xi\frac{b_i - a_i}{2}\right) \tag{1.16}$$

Substitution of the expansion (1.16) in Eqs. (1.6) and change of the limits of integration and order of summation give us

$$(\pi/4)\sum_{i=1}^{n}(b_i^2 - a_i^2)\sum_{n=0}^{\infty} c_n^{(i)}\int_0^{\infty}(1 - G(\xi))J_0(\xi r)J_n\left(\xi\frac{b_i + a_i}{2}\right)J_n\left(\xi\frac{b_i - a_i}{2}\right)d\xi = f_j(r)$$

$$(j = 1, 2, \ldots, N), \quad (a_j < r < b_j) \tag{1.17}$$

Next in accordance with the expansion of the functions $f_j(r)$ into a series

$$f_j(r) = (1/2) f_0^{(j)} + \sum_{k=1}^{\infty} f_k^{(j)} \cos k\varphi_j, \quad (0 < \varphi_j < \pi) \tag{1.18}$$

and account of the addition formula (1.15) on the base of (1.17) we get the following infinite system of algebraic equations for determination of the coefficients $c_n^{(i)}$ $(n = 1, 2, \ldots ; i = 1, 2, \ldots, N)$

$$\sum_{i=1}^{N}(b_i^2-a_i^2)\sum_{n=0}^{\infty}c_n^{(i)}\Omega_{nk}^{(ij)}=(2/\pi)f_k^{(j)},\ (k=0,1,2,...\ ;\ j=1,2,...,N) \tag{1.19}$$

where the coefficients $\Omega_{nk}^{(ij)}$ can be obtained using the formula

$$\Omega_{nk}^{(ij)}=\int_0^{\infty}(1-G(\xi))J_n\left(\xi\frac{b_i+a_i}{2}\right)J_n\left(\xi\frac{b_i-a_i}{2}\right)J_k\left(\xi\frac{b_j+a_j}{2}\right)J_k\left(\xi\frac{b_j-a_j}{2}\right)d\xi \tag{1.20}$$

$$(n,\ k=0,1,2,...\ ;\ i,j=1,2,...,N)$$

and as satisfying the condition of symmetry

$$\Omega_{nk}^{(ij)}=\Omega_{kn}^{(ji)} \tag{1.21}$$

Thus the solution of the integral Eqs. (1.6), (1.7) reduces to the solution of infinite systems of algebraic Eqs. (1.19) according to unknown coefficients $c_n^{(i)}\ (n=0,1,2,...),\ (i=1,2,...,N)$.

In the particular case of $N=1,\ a_1=a,\ b_1=b$ the system (1.19) takes the form

$$\sum_{n=0}^{\infty}c_n\Omega_{nk}=\frac{2}{\pi(b^2-a^2)}f_k,\ (k=0,1,2,...) \tag{1.22}$$

where

$$c_n=c_n^1,\quad f_k=f_k^1$$

$$\Omega_{nk}=\frac{2}{(b+a)}\int_0^{\infty}(1-\tilde{G}(\eta))J_n(\eta)J_n(\alpha\eta)J_k(\eta)J_k(\alpha\eta)d\eta \tag{1.23}$$

$$\tilde{G}(\eta)=G\left(\frac{2\eta}{b+a}\right),\quad \alpha=(b-a)/(b+a)$$

Further simplification of the system is possible for the solution of boundary-value problem for a half-space $r\geq 0,\ 0<r<\infty$. In this case the axisymmetric solution of the Laplace equation in the domain $z>0,\ r\geq 0$ is written in the form:

$$u(r,z)=\int_0^{\infty}C(\xi)\exp(-\xi z)J_0(\xi r)d\xi \tag{1.24}$$

and mixed boundary conditions in the plane $z=0$

$$u(r,0)=f_i(r),\quad a_i<r<b_i,\ i=1,2,...,N \tag{1.25}$$

$$\left.\frac{\partial u(r,z)}{\partial z}\right|_{z=0}=0,\quad 0<r<a_1,\ b_i<r<a_{i+1},\ i=1,2,...,N-1,\ r>b_N \tag{1.26}$$

reduce to the following integral equations for determination of the function $C(\xi)$

$$\begin{cases} \int_0^\infty C(\xi) J_0(\xi r) d\xi = f_i(r), \quad a_i < r < b_i, \quad i = 1,2,\ldots,N \\ \int_0^\infty \xi C(\xi) J_0(\xi r) d\xi = 0, \quad 0 < r < a_1, \quad b_i < r < a_{i+1}, \quad i = 1,2,\ldots,N-1, \quad r > b_N \end{cases} \tag{1.27}$$

In the case of $N = 1$ $(a_1 = a,\ b_1 = b)$ the solution of Eqs. (1.27) takes the form

$$C(\xi)(\pi/4)(b^2 - a^2)\sum_{n=0}^{\infty} c_n J_n\left(\xi\frac{b+a}{2}\right) J_n\left(\xi\frac{b-a}{2}\right) \tag{1.28}$$

where coefficients c_n $(n = 0,1,2,\ldots)$ can be obtained using the solution of infinite system of algebraic equations

$$\sum_{n=0}^{\infty} c_n \omega_{nk} = \frac{2}{\pi(b^2 - a^2)} f_k \quad (k = 0,1,2,\ldots) \tag{1.29}$$

Here

$$\omega_{nk} = (2/(b+a))\int_0^\infty J_n(\eta) J_n(\alpha\eta) J_k(\eta) J_k(\alpha\eta) d\eta \quad (n,\ k = 0,1,2,\ldots) \tag{1.30}$$

2. Let us analyse the coefficients ω_{nk} determined using the formula (1.30) in the form of the improper integral of the product of four Bessel functions. The simplest approximate numerical method for these coefficients is based on using the representation of ω_{nk} in the form

$$\omega_{nk} = \frac{2}{b+a}\int_0^{\eta_0} J_n(\eta) J_n(\alpha\eta) J_k(\eta) J_k(\alpha\eta) d\eta + \frac{2}{b+a}\int_{\eta_0}^{\infty} J_n(\eta) J_n(\alpha\eta) J_k(\eta) J_k(\alpha\eta) d\eta \tag{2.1}$$

where the reason for choice of the value η_0 is the possibility of using in the last integral asymptotic formulas for the Bessel functions

$$J_n(\eta) \cong \sqrt{\frac{2}{\pi\eta}} \cos\left[\eta - \frac{\pi n}{2} - \frac{\pi}{4}\right], \quad (\eta \geq \eta_0) \tag{2.2}$$

The asymptotic expression

$$\begin{aligned} J_n(\eta) J_k(\eta) J_n(\alpha\eta) J_k(\alpha\eta) &\cong (1/(2\pi^2\eta^2\alpha))[1 + (-1)^{n-k} + ((-1)^n + \\ &+ (-1)^k)(\sin 2\eta + \sin 2\alpha\eta) + \cos(2(1-\alpha)\eta) - (-1)^{n+k}\cos(2(1+\alpha)\eta)] \end{aligned} \tag{2.3}$$

is correct with account of (2.2). Using (2.3) we obtain the following form for the second integral in the right part of (2.1)

$$\begin{aligned} &\int_{\eta_0}^{\infty} J_n(\eta) J_k(\eta) J_n(\alpha\eta) J_k(\alpha\eta) d\eta \cong (1/(2\pi^2\alpha\eta_0))[1 + (-1)^{n-k} + ((-1)^n + \\ &+ (-1)^k)(\sin 2\eta_0 + \sin 2\alpha\eta_0 - 2\eta_0 Ci(2\eta_0) - 2\alpha\eta_0 Ci(2\alpha\eta_0)) + \\ &+ \cos(2(1-\alpha)\eta_0) - (-1)^{n+k}\cos(2(1+\alpha)\eta_0) + 2(1-\alpha)\eta_0 si(2(1-\alpha)\eta_0) - \\ &- (-1)^{n+k} 2(1+\alpha)\eta_0 si(2(1+\alpha)\eta_0)] \end{aligned} \tag{2.4}$$

Here

$$si(z) = -\int_z^\infty \frac{\sin\xi}{\xi}\,d\xi, \quad Ci(z) = -\int_z^\infty \frac{\cos\xi}{\xi}\,d\xi$$

The first integral in (2.1) may be calculated numerically. Let as show another method of reduction of the integral ω_{nk} to the form, which is convenient for numerical calculations.

Using the formula

$$J_n(\eta)J_n(\alpha\eta) = (1/\pi)\int_0^\pi J_0\left(\eta\sqrt{1+\alpha^2-2\alpha\cos\varphi}\right)\cos n\varphi\,d\varphi \tag{2.5}$$

we get the following expression for ω_{nk} :

$$\omega_{nk}\left(\frac{b+a}{2}\right) = \frac{1}{\pi^2}\int_0^\pi \cos n\varphi \int_0^\pi \cos k\theta\left(\int_0^\infty J_0(\eta\rho(\varphi))J_0(\eta\rho(\theta))\,d\eta\right)d\theta\,d\varphi \tag{2.6}$$

where $\rho(\varphi) = \sqrt{1+\alpha^2-2\alpha\cos\varphi}$

The value of inside integral in (2.6) is known

$$\int_0^\infty J_0(\eta\rho(\varphi))J_0(\eta\rho(\theta))\,d\eta = \frac{2}{\pi}\begin{cases}\dfrac{1}{\rho(\theta)}\mathbf{K}\left(\dfrac{\rho(\varphi)}{\rho(\theta)}\right), & \theta > \varphi \\[2ex] \dfrac{1}{\rho(\varphi)}\mathbf{K}\left(\dfrac{\rho(\theta)}{\rho(\varphi)}\right), & \theta < \varphi\end{cases} \tag{2.7}$$

Here $\mathbf{K}(z)$ is the complete elliptic integral of the second kind.

Then we obtain the representation of coefficients ω_{nk} in the form of multiple integrals between finite limits

$$\begin{aligned}\left(\frac{b+a}{2}\right)\omega_{nk} &= \frac{2}{\pi^3}\int_0^\pi \frac{\cos n\varphi}{\rho(\varphi)}\int_0^\varphi \cos k\theta\,\mathbf{K}\left(\frac{\rho(\theta)}{\rho(\varphi)}\right)d\theta\,d\varphi + \\ &+ \frac{2}{\pi^3}\int_0^\pi \frac{\cos k\varphi}{\rho(\varphi)}\int_0^\varphi \cos n\theta\,\mathbf{K}\left(\frac{\rho(\theta)}{\rho(\varphi)}\right)d\theta\,d\varphi\end{aligned} \tag{2.8}$$

Since the function $\mathbf{K}(z)$ has singularity if $z \to 1$, then inside integrals in (2.8) should be reduced using the relationship

$$\mathbf{K}(z) = \ln\left(4/\sqrt{1-z^2}\right) + O\left((1-z^2)\ln\sqrt{1-z^2}\right), \quad z \to 1 \tag{2.9}$$

In consequence we get

$$\left(\frac{b+a}{2}\right)\omega_{nk} = \frac{2}{\pi^3}\int_0^\pi \frac{\cos n\varphi}{\rho(\varphi)}\int_0^\varphi \cos k\theta\left[\mathbf{K}\left(\frac{\rho(\theta)}{\rho(\varphi)}\right) - \ln\left(\frac{4\rho(\varphi)}{\sqrt{2\alpha}\sqrt{\cos\theta-\cos\varphi}}\right)\right]d\theta\,d\varphi +$$

$$+\frac{2}{\pi^3}\int_0^\pi \frac{\cos k\varphi}{\rho(\varphi)}\int_0^\varphi \cos n\theta\left[\mathbf{K}\left(\frac{\rho(\theta)}{\rho(\varphi)}\right) - \ln\left(\frac{4\rho(\varphi)}{\sqrt{2\alpha}\sqrt{\cos\theta-\cos\varphi}}\right)\right]d\theta\,d\varphi +$$

$$
+\frac{2}{\pi^3 k}\int_0^{\pi}\frac{\cos n\varphi \sin k\varphi}{\rho(\varphi)}\ln\left(\frac{4}{\sqrt{2\alpha}}\rho(\varphi)\right)d\varphi+\frac{2}{\pi^3 n}\int_0^{\pi}\frac{\cos k\varphi \sin n\varphi}{\rho(\varphi)}\ln\left(\frac{4}{\sqrt{2\alpha}}\rho(\varphi)\right)d\varphi-
$$

$$
-\frac{1}{\pi^3}\int_0^{\pi}\frac{\cos n\varphi}{\rho(\varphi)}\int_0^{\varphi}\cos k\theta \ln|\cos\theta-\cos\varphi|d\theta d\varphi-
$$

$$
-\frac{1}{\pi^3}\int_0^{\pi}\frac{\cos k\varphi}{\rho(\varphi)}\int_0^{\varphi}\cos n\theta \ln|\cos\theta-\cos\varphi|d\theta d\varphi \tag{2.10}
$$

Finally using the known expansion

$$
\ln|2(\cos\theta-\cos\varphi)|=-2\sum_{m=1}^{\infty}\frac{\cos m\theta\cos m\varphi}{m} \tag{2.11}
$$

the inside integrals in the last terms of (2.10) can be represented as series

$$
\int_0^{\varphi}\cos n\theta \ln|\cos\theta-\cos\varphi|d\theta=-\frac{\sin n\varphi}{n}\ln 2-\varphi\frac{\cos n\varphi}{n}-
$$

$$
-\sum_{m=1}^{n-1}\frac{\sin(m-n)\varphi\cos m\varphi}{m(n-m)}-2\sum_{m=1}^{\infty}\frac{\sin m\varphi\cos(m+n)\varphi}{m(m+n)},\quad n=1,2,... \tag{2.12}
$$

$$
\int_0^{\varphi}\ln|\cos\theta-\cos\varphi|\,d\theta=-\varphi\ln 2-2\sum_{m=1}^{\infty}\frac{\sin m\varphi\cos m\varphi}{m^2} \tag{2.13}
$$

Let as note that the solution of the system (1.29) permits to determine the function $C(\xi)$ satisfied the integral Eqs. (1.27) if $N=1$ using the formula (1.28) and, besides, to obtain the value

$$
\left(\int_0^{\infty}\xi C(\xi)J_0(\xi r)d\xi\right)_{a<r<b}=\frac{b^2-a^2}{2\sqrt{(b^2-r^2)(r^2-a^2)}}\sum_{n=0}^{\infty}(-1)^n c_n T_{2n}\left(\sqrt{\frac{r^2-a^2}{b^2-a^2}}\right) \tag{2.14}
$$

where $T_{2n}(z)$ are Chebishev polynomials of the first kind. The value (2.14) is important for the analysis of concrete boundary-value problem.

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