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ЗВЕСТИЯ АКАДЕМИИ НАУК АРМЯНСКОЙ ССР
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ON THE STABILITY OF THE MOTION OF A ROTATING SHAFT BEARING AN UNSYMMETRIC ROTOR

1. *Introduction*. The motion of a rotating shaft with a symmetrical ross-section, bearing a symmetrical body (e. g. a disk) has been invesigated by various investigators. On the contrary, authors were unable o find publications about the case that the body is unsymmetrical, ith the exception of two articles of Crandall [1, 2] who, however, nly investigated the case in which the mass centre of the body is ixed, the body being only able to rotate under the action of restoring noments.

In the present paper the more general case in which the mass entre of the body is free to move with the shaft, is treated. We supose that the mass centre is situated on the axis of revolution of the haft and that one of the main axes of inertia coincides with this axis when the shaft is undeformed. For the investigation analogous methods ave been used as in a previous study of the motion of a symmetrical shaft, bearing a symmetrical body and loaded by an axial force [3].

After the investigation had been largely finished, authors became aware of a couple of publications of Aiba [4, 5], who investigated the same problem and found analogous results. However, authors thought that it would be worth-while to publish the results of their own work, because in a sense the treatment was more general and, moreover, permits to make a comparison with one of the general enunciations of Četaev on the stability of motion.

2. *Description of the motion of the system*. Although the equations of motion which we intend to derive will be valid for any system of supports of the rotating shaft, we restrict ourselves for the time being to the case of a cantilever shaft, represented in fig. 1.

The shaft is "clamped" at the origin ω of the fixed coordinate system (ω, ξ, η, ζ), its axis of revolution coinciding with the axis $\omega\zeta$ in the undeformed state. In the deformed state ω_0 is the elastic line of the shaft, o then being the mass centre of the body which is attached to the shaft. One of the main axes of inertia of the body is oz; in the undeformed position this axis coincides with $\omega\zeta$. The two other main axes of inertia are ox and oy.

The translation and rotation of the body are described by means of the two auxiliary systems (o, ξ^*, η^*, ζ^*) and (o, x^*, y^*, z), shown in the figure. The first system indicates the translation of the body, whereas the second one describes the bending of the shaft. The rotation of shaft and body is given by the angle ψ between $(o,\ x,\ y)$ and $(o,\ x^*,\ y^*)$. The rotation of the system $(o,\ x^*,\ y^*,\ z)$ will be described by means of Rodrigues coordinates [6, 7].

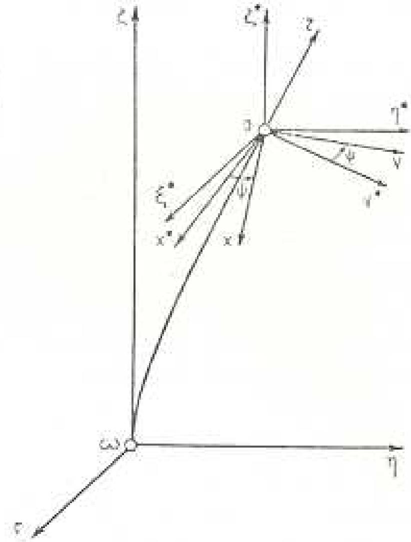


Fig. 1. The shaft in a deflected position.

We will indicate column vectors by means of lower case letters on top of which a bar has been printed, and matrices in the same way but with capital letters. The transposition operation is indicated by a tilde. Thus, a row vector is represented by a lower case letter in top of which a bar and a tilde have been printed.

We then can define the following vectors:

$$\tilde{\bar{\rho}} = (\xi,\ \eta,\ \zeta),\quad \tilde{\bar{\rho}}_0 = (\xi_0,\ \eta_0,\ \zeta_0),\quad \tilde{\bar{\rho}}^* = (\xi^*,\ \eta^*,\ \zeta^*) \tag{2.1}$$

$$\tilde{\bar{r}} = (x,\ y,\ z),\quad \tilde{\bar{r}}^* = (x^*,\ y^*,\ z)$$

Here $\bar{\rho}_0$ is the radius vector of the mass centre o. The transformation between the various vectors can be represented by the formulae

$$\bar{\rho} = \bar{\rho}_0 + \bar{\rho}^*,\quad \bar{\rho}^* = \bar{G}^* \bar{r}^*,\quad \bar{r}^* = \bar{G}^{**} \bar{r} \tag{2.2}$$

where the rotation matrices $\bar{G}^*$ and $\bar{G}^{**}$ are given by

$$\bar{G}^* = \begin{vmatrix} 1 - \frac{1}{2}\chi^{*2} & \frac{1}{2}\varphi^*\chi^* & \chi^* \\ \frac{1}{2}\varphi^*\chi^* & 1 - \frac{1}{2}\varphi^{*2} & -\varphi^* \\ -\chi^* & \varphi^* & 1 - \frac{1}{2}(\varphi^{*2} + \chi^{*2}) \end{vmatrix} \tag{2.3a}$$

$$\overline{G}^{**} = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.3b}$$

respectively. Here φ^* and χ^* are Rodrigues coordinates; in first approximation, φ^* is the rotation of the system (o, x^*, y^*, z) around the axis ox^* and χ^* the rotation around the axis oy^*. In the formula (2.3a) for $\overline{G}^*$ terms of third and higher degree in φ^* and χ^* have been omitted. The product of $\overline{G}^*$ and $\overline{G}^{**}$ is called $\overline{G}$:

$$\overline{G} = \overline{G}^*\overline{G}^{**} \tag{2.4}$$

so that (1) and (2) yield

$$\overline{\rho} = \overline{\rho}_0 + \overline{G}\overline{r} \tag{2.5}$$

The components with respect to the fixed coordinate system of the absolute velocity of a point of the body are given by the vector

$$\overline{v}^* = \dot{\overline{\rho}} = \dot{\overline{\rho}}_0 + \dot{\overline{G}}\overline{r} \tag{2.6}$$

Now

$$\overline{\rho} - \overline{\rho}_0 = \overline{G}\overline{r}, \quad \overline{r} = \widetilde{\overline{G}}(\overline{\rho} - \overline{\rho}_0) \tag{2.7}$$

(because $\widetilde{\overline{G}} = \overline{G}^{-1}$), so that also

$$\overline{v}^* = \dot{\overline{\rho}}_0 + \overline{\Omega}^*(\overline{\rho} - \overline{\rho}_0) \tag{2.8}$$

where

$$\overline{\Omega}^* = \dot{\overline{G}}\widetilde{\overline{G}} \tag{2.9}$$

We find

$$\overline{\Omega}^* = \begin{pmatrix} 0 & -\omega_\zeta & \omega_\eta \\ \omega_\zeta & 0 & -\omega_\xi \\ -\omega_\eta & \omega_\xi & 0 \end{pmatrix} \tag{2.10}$$

with

$$\begin{gathered} \omega_\xi = \dot{\varphi}^* + \chi^*\dot{\psi}, \quad \omega_\eta = \dot{\chi}^* - \varphi^*\dot{\psi} \\ \omega_\zeta = \frac{1}{2}(\varphi^*\dot{\chi}^* - \chi^*\dot{\varphi}^*) + \left(1 - \frac{1}{2}\varphi^{*2} - \frac{1}{2}\chi^{*2}\right)\dot{\psi} \end{gathered} \tag{2.11}$$

Thus instead of (2.8) we can write:

$$\overline{v}^* = \dot{\overline{\rho}}_0 + \overline{\omega}^*(\overline{\rho} - \overline{\rho}_0) \tag{2.12}$$

the vector of the angular velocity being defined by

$$\widetilde{\overline{\omega}}^* = (\omega_\xi, \omega_\eta, \omega_\zeta) \tag{2.13}$$

More useful are the components of the angular velocity vector with respect to the rotating system (o, x, y, z). By means of (2.6) we have for the components with respect to this system of the absolute velocity of a point of the body:

$$\overline{v} = \tilde{\overline{G}}\overline{v}^* = \tilde{\overline{G}}\dot{\overline{\rho}}_0 + \overline{\Omega}\overline{r} \tag{2.14}$$

with

$$\overline{\Omega} = \tilde{\overline{G}}\dot{\overline{G}} \tag{2.15}$$

Now we find

$$\overline{\Omega} = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_x & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix} \tag{2.16}$$

with

$$\omega_x = \dot{\varphi}^* \cos\psi + \dot{\chi}^* \sin\psi, \quad \omega_y = -\dot{\varphi}^* \sin\psi + \dot{\chi}^* \cos\psi$$
$$\omega_z = \dot{\psi} + \frac{1}{2}(\chi^* \dot{\varphi}^* - \varphi^* \dot{\chi}^*) \tag{2.17}$$

so that also

$$\overline{v} = \tilde{\overline{G}}\dot{\overline{\rho}}_0 + \overline{\omega}\overline{r} \tag{2.18}$$

here the vector of the angular velocity reads

$$\overline{\omega} = (\omega_x, \omega_y, \omega_z) \tag{2.19}$$

We easily verify the relation

$$\overline{\omega}^* = \overline{G}\overline{\omega} \tag{2.20}$$

The velocity of the mass centre can best be determined by means of (2.6), by putting $\overline{r} = \overline{o}$:

$$\overline{v}_0^* = \dot{\overline{\rho}}_0 \tag{2.21}$$

so that

$$\overline{v}_0^* = (\dot{\xi}_0, \dot{\eta}_0, \dot{\zeta}_0) \tag{2.22}$$

Here

$$\dot{\zeta}_0 = 0 \tag{2.23}$$

3. *Derivation of the equations of motions by means of energy methods (Lagrange).* In using the method of Lagrange's equations we first have to find the expression for the potential energy U. Because the shaft is rotationally symmetric, the relations between the forces

and moments applying on the shaft and the displacement quantities can be written, according to fig. 2, as

$$\begin{aligned} -F_\xi &= c_{11}\xi_0 + c_{12}\chi^*, & -M_\eta &= c_{21}\xi_0 + c_{22}\chi^* \\ -F_\eta &= c_{11}\eta_0 - c_{12}\varphi^*, & M_\xi &= c_{21}\eta_0 - c_{22}\varphi^* \end{aligned} \tag{3.1}$$

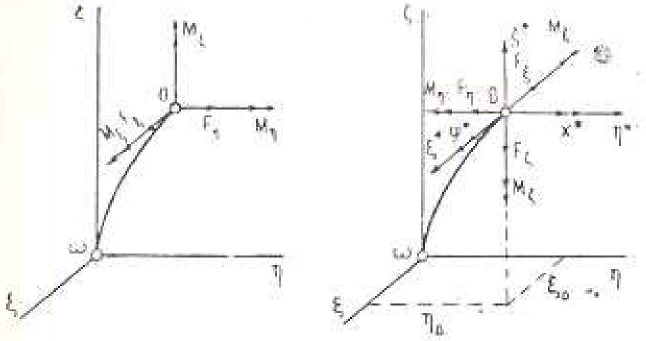


Fig. 2. The forces exerted by the shaft on the body, the forces exerted by the body on the shaft and the rotations φ^* and χ^*.

For a cantilever shaft with the length a we have, for example,

$$c_{11} = \frac{12EJ}{a^3}, \qquad c_{12} = c_{21} = -\frac{6EJ}{a^2}, \qquad c_{22} = \frac{4EJ}{a} \tag{3.2}$$

Now owing to Clapeyron's law we find

$$U = -\frac{1}{2}(F_\xi \xi_0 + F_\eta \eta_0 + M_\xi \varphi^* + M_\eta \chi^*) \tag{3.3}$$

and together with (1) we obtain the expression for the potential energy

$$U = \frac{1}{2} c_{11}(\xi_0^2 + \eta_0^2) + c_{12}(\xi_0 \chi^* - \eta_0 \varphi^*) + \frac{1}{2} c_{22}(\varphi^{*2} + \chi^{*2}) \tag{3.4}$$

Instead of the coordinates ξ_0, η_0, φ^* and χ^*, related to the fixed system, we henceforth will use the coordinates

$$\begin{aligned} x &= \xi_0 \cos\psi + \eta_0 \sin\psi, & y &= -\xi_0 \sin\psi + \eta_0 \cos\psi \\ \varphi &= \varphi^* \cos\psi + \chi^* \sin\psi, & \chi &= -\varphi^* \sin\psi + \chi^* \cos\psi \end{aligned} \tag{3.5}$$

which belong to the rotating system. In the new coordinates the expression for the potential energy (3.4) reads

$$U = \frac{1}{2} c_{11}(x^2 + y^2) + c_{12}(x\chi - y\varphi) + \frac{1}{2} c_{22}(\varphi^2 + \chi^2) \tag{3.6}$$

The kinetic energy of the system is given by the expression

$$T = \frac{1}{2} m (v_{0\xi}^2 + v_{0\eta}^2 + v_{0\zeta}^2) + \frac{1}{2}(J_x \omega_x^2 + J_y \omega_y^2 + J_z \omega_z^2) \tag{3.7}$$

where $v_{0\xi}$ etc. are the elements of the vector $\overline{v_0^*}$ (2.22) and ω_x, ω_y, ω_z are determined by (2.17). We first reduce ξ_0, η_0, φ^*, χ^* to x, y, φ, χ. Inverting (3.5) we have

$$\begin{aligned} \xi_0 &= x\cos\psi - y\sin\psi, \quad \eta_0 = x\sin\psi + y\cos\psi \\ \varphi^* &= \varphi\cos\psi - \chi\sin\psi, \quad \chi^* = \varphi\sin\psi + \chi\cos\psi \end{aligned} \tag{3.8}$$

so that

$$\begin{aligned} \dot{\xi}_0 &= \dot{x}\cos\psi - \dot{y}\sin\psi - \dot{\psi}(x\sin\psi + y\cos\psi) \\ \dot{\eta}_0 &= \dot{x}\sin\psi + \dot{y}\cos\psi + \dot{\psi}(x\cos\psi - y\sin\psi) \\ \dot{\varphi}^* &= \dot{\varphi}\cos\psi - \dot{\chi}\sin\psi - \dot{\psi}(\varphi\sin\psi + \chi\cos\psi) \\ \dot{\chi}^* &= \dot{\varphi}\sin\psi + \dot{\chi}\cos\psi + \dot{\psi}(\varphi\cos\psi + \chi\sin\psi) \end{aligned} \tag{3.9}$$

Thus the velocity components read

$$\begin{aligned} v_{0\xi} &= \dot{x}\cos\psi - \dot{y}\sin\psi - \dot{\psi}(x\sin\psi + y\cos\psi) \\ v_{0\eta} &= \dot{x}\sin\psi + \dot{y}\cos\psi + \dot{\psi}(x\cos\psi - y\sin\psi) \\ v_{0\zeta} &= 0 \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} \omega_x &= \dot{\varphi} - \chi\dot{\psi}, \qquad \omega_y = \dot{\chi} + \varphi\dot{\psi} \\ \omega_z &= \dot{\psi}\left(1 - \frac{1}{2}\varphi^2 - \frac{1}{2}\chi^2\right) + \frac{1}{2}(\chi\dot{\varphi} - \varphi\dot{\chi}) \end{aligned} \tag{3.11}$$

respectively. Inserting these expressions into (3.7) we obtain

$$\begin{aligned} T = \frac{1}{2}m[\dot{x}^2 + \dot{y}^2 + 2\dot{\psi}(x\dot{y} - y\dot{x}) + \dot{\psi}^2(x^2 + y^2)] + \\ + \frac{1}{2}J_x(\dot{\varphi} - \chi\dot{\psi})^2 + \frac{1}{2}J_y(\dot{\chi} + \varphi\dot{\psi})^2 + \\ + \frac{1}{2}J_z[\dot{\psi}^2(1 - \varphi^2 - \chi^2) + \dot{\psi}(\chi\dot{\varphi} - \varphi\dot{\chi})] \end{aligned} \tag{3.12}$$

The shaft is driven by a moment M around the axis 0ζ. The generalised forces related to it are found by determining the virtual work

$$\delta W = M\delta\theta_\zeta \tag{3.13}$$

Now $\delta\theta_\zeta$ can best be calculated by considering the expression (2.11) for $\omega_\zeta = \dot{\theta}_\zeta$, by replacing the quantities φ^* and χ^* in the expression by φ and χ, according to (3.8) and (3.9):

$$\omega_\zeta = \frac{1}{2}(\varphi\dot{\chi} - \dot{\varphi}\chi) + \dot{\psi} \tag{3.14}$$

and by replacing ω_ζ, $\dot{\varphi}$, $\dot{\chi}$ and $\dot{\psi}$ by $\delta\theta_\zeta$, $\delta\theta$, $\delta\chi$ and $\delta\psi$ respectively:

$$\delta\theta_z = \frac{1}{2}(\varphi\delta\chi - \chi\delta\varphi) + \delta\psi \tag{3.15}$$

In combination with (3.13) we then find for the various generalised forces:

$$Q_x = 0, \quad Q_y = 0, \quad Q_\varphi = -\frac{1}{2}M\chi, \quad Q_\chi = \frac{1}{2}M\varphi, \quad Q_\psi = M \tag{3.16}$$

We now are able to derive the equations of motion by means of the formula

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} = Q_i \quad (i = 1, \ldots, n) \tag{3.17}$$

Owing to (3.6), (3.12) and (3.16) we obtain

$$m(\ddot{x} - y\ddot{\psi} - 2\dot{y}\dot{\psi} - x\dot{\psi}^2) + c_{11}x + c_{12}\chi = 0 \tag{3.18a}$$

$$m(\ddot{y} + x\ddot{\psi} + 2\dot{x}\dot{\psi} - y\dot{\psi}^2) + c_{11}y - c_{12}\varphi = 0 \tag{3.18b}$$

$$J_x(\ddot{\varphi} - \chi\ddot{\psi} - \dot{\chi}\dot{\psi}) - J_y(\dot{\chi}\dot{\psi} + \varphi\dot{\psi}^2) +$$
$$+ J_z\left(\frac{1}{2}\chi\ddot{\psi} + \dot{\chi}\dot{\psi} + \varphi\dot{\psi}^2\right) - c_{12}y + c_{22}\varphi = -\frac{1}{2}M\chi \tag{3.18c}$$

$$J_x(\dot{\varphi}\dot{\psi} - \chi\dot{\psi}^2) + J_y(\ddot{\chi} + \varphi\ddot{\psi} + \dot{\varphi}\dot{\psi}) +$$
$$+ J_z\left(-\frac{1}{2}\varphi\ddot{\psi} - \dot{\varphi}\dot{\psi} + \chi\dot{\psi}^2\right) + c_{12}x + c_{22}\chi = \frac{1}{2}M\varphi \tag{3.18d}$$

$$m\{x\ddot{y} - y\ddot{x} + \ddot{\psi}(x^2 + y^2) + 2\dot{\psi}(x\dot{x} + y\dot{y})\} -$$
$$- J_x\{\chi(\ddot{\varphi} - \chi\ddot{\psi} - \dot{\chi}\dot{\psi}) + \dot{\chi}(\dot{\varphi} - \chi\dot{\psi})\} +$$
$$+ J_y\{\varphi(\ddot{\chi} + \varphi\ddot{\psi} + \dot{\varphi}\dot{\psi}) + \dot{\varphi}(\dot{\chi} + \varphi\dot{\psi})\} +$$
$$+ J_z\left\{\ddot{\psi} + \frac{1}{2}(\chi\ddot{\varphi} - \varphi\ddot{\chi}) - \ddot{\psi}(\varphi^2 + \chi^2) - 2\dot{\psi}(\varphi\dot{\varphi} + \chi\dot{\chi})\right\} = M \tag{3.18e}$$

The equation (3.18e) can be simplified by combining it with the four other equations of motion, (3.18a)—(3.18d). Then we find

$$M = -(J_x - J_y)(\dot{\varphi} - \chi\dot{\psi})(\dot{\chi} + \varphi\dot{\psi}) +$$
$$+ J_z\left\{\ddot{\psi} - \dot{\psi}(\varphi\dot{\varphi} + \chi\dot{\chi}) + \frac{1}{2}(\chi\ddot{\varphi} - \varphi\ddot{\chi})\right\} \tag{3.19}$$

Adding $-\frac{1}{2}\chi$ times M (3.19) to equation (3.18c) and $\frac{1}{2}\varphi$ times M (3.19) to equation (3.18d) we find for the four first equations of motion

$$
\begin{gathered}
m(\ddot{x} - y\ddot{\psi} - 2\dot{y}\dot{\psi} - x\dot{\psi}^2) + c_{11}x + c_{12}\chi = 0 \\
m(\ddot{y} + x\ddot{\psi} + 2\dot{x}\dot{\psi} - y\dot{\psi}^2) + c_{11}y - c_{12}\varphi = 0 \\
J_x(\ddot{\varphi} - \chi\ddot{\psi} - \dot{\chi}\dot{\psi}) - (J_y - J_z)\dot{\psi}(\dot{\chi} + \varphi\dot{\psi}) - c_{12}y + c_{22}\varphi = -M\chi \qquad (3.20) \\
J_y(\ddot{\chi} + \varphi\ddot{\psi} + \dot{\varphi}\dot{\psi}) + (J_x - J_z)\dot{\psi}(\dot{\varphi} - \chi\dot{\psi}) + c_{12}x + c_{22}\chi = M\varphi
\end{gathered}
$$

In sections 5—7 only the motion will be considered for the case $\dot{\psi} = \omega = \text{const}$. Then M (3.19) is of second order in the remaining four displacement quantities, so that the four first equations of motion reduce to

$$
\begin{gathered}
m(\ddot{x} - 2\omega\dot{y} - \omega^2 x) + c_{11}x + c_{12}\chi = 0 \\
m(\ddot{y} + 2\omega\dot{x} - \omega^2 y) + c_{11}y - c_{12}\varphi = 0 \\
J_x(\ddot{\varphi} - \omega\dot{\chi}) - (J_y - J_z)\omega(\dot{\chi} + \omega\varphi) - c_{12}y + c_{22}\varphi = 0 \qquad (3.21) \\
J_y(\ddot{\chi} + \omega\dot{\varphi}) + (J_x - J_z)\omega(\dot{\varphi} - \omega\chi) + c_{12}x + c_{22}\chi = 0
\end{gathered}
$$

Before we shall discuss these equations in detail, we will show how the equations of motion can be found by means of the fundamental law of analytical mechanics.

4. *Derivation of the equations of motion by means of the fundamental law of analytical mechanics (Newton and Euler).* The motion of the mass centre o of the body is determined by Newton's law:

$$
\begin{Bmatrix} F_x \\ F_y \\ F_z \end{Bmatrix} = m\overline{a}_0 \qquad (4.1)
$$

F_x, F_y and F_z being the components in the directions ox, oy, oz of the force which the shaft exerts on the body, and $\overline{a}_0$ being the vector with respect to the system (o, x, y, z) of the absolute acceleration of the mass centre.

For the forces the relation

$$
\begin{Bmatrix} F_x \\ F_y \\ F_z \end{Bmatrix} = \tilde{G} \begin{Bmatrix} F_\xi \\ F_\eta \\ F_\zeta \end{Bmatrix} \qquad (4.2)
$$

holds. By means of the relations (3.1) for the forces F_ξ and F_η and the relations (2.4), (2.3a) and (2.3b) we find, restricting ourselves to quantities of zero and first order:

$$
F_x = -(c_{11}x + c_{12}\chi), \quad F_y = -(c_{11}y - c_{12}\varphi) \qquad (4.3)
$$

For the acceleration $\overline{a}_0$ the relation

$$
\overline{a}_0 = \dot{\overline{v}}_0 + \overline{\omega} \times \overline{v}_0 \qquad (4.4)
$$

holds, where $\bar{v}_0$ is the vector with respect to the system (o, x, y, z) of the velocity of the mass centre. We have

$$\bar{v}_0 = \tilde{G}\,\bar{v}_0^* \tag{4.5}$$

the components of $\bar{v}_0^*$ being given by (3.10). Omitting again all quantities of second and higher order, we have, because of (2.4), (2.3a) and (2.3b),

$$\bar{v}_0 = \begin{vmatrix} \dot{x} - \dot{\psi} y \\ \dot{y} + \dot{\psi} x \\ 0 \end{vmatrix} \tag{4.6}$$

So (4.1), together with (4.4), (4.6), (3.11) and (4.3) gives for the first two equations of motion

$$\begin{aligned} m(\ddot{x} - \ddot{\psi} y - 2\dot{\psi}\dot{y} - \dot{\psi}^2 x) + c_{11} x + c_{12}\chi = 0 \\ m(\ddot{y} + \ddot{\psi} x + 2\dot{\psi}\dot{x} - \dot{\psi}^2 y) + c_{11} y - c_{12}\varphi = 0 \end{aligned} \tag{4.7}$$

in agreement with the first two equations (3.20).

Euler's equations read

$$\begin{aligned} M_x &= J_x\dot{\omega}_x - (J_y - J_z)\,\omega_y\omega_z \\ M_y &= J_y\dot{\omega}_y - (J_z - J_x)\,\omega_z\omega_x \\ M_z &= J_z\dot{\omega}_z - (J_x - J_y)\,\omega_x\omega_y \end{aligned} \tag{4.8}$$

the moments being determined by

$$\begin{pmatrix} M_x \\ M_y \\ M_z \end{pmatrix} = \tilde{G}\begin{pmatrix} M_\xi \\ M_\eta \\ M_\zeta \end{pmatrix} \tag{4.9}$$

Now M_ξ and M_η are given by (3.1), whereas M_ζ follows from the shaft equilibrium around the axis $\omega\zeta$ and relations (3.8):

$$M_\zeta = M + F_\xi\eta_0 - F_\eta\xi_0 = M - c_{12}(\xi_0\varphi^* + \eta_0\chi^*) = M - c_{12}(x\varphi + y\chi) \tag{4.10}$$

Altogether we have

$$\begin{aligned} M_x = c_{12} y - c_{22}\varphi - M\chi, \quad M_y = -c_{12} x - c_{22}\chi + M\varphi \\ M_z = M\left\{1 - \frac{1}{2}(\varphi^2 + \chi^2)\right\} \end{aligned} \tag{4.11}$$

Euler's equations (4.8) now become, with (3.11):

$$J_x(\ddot{\varphi} - \chi\ddot{\psi} - \dot{\chi}\dot{\psi}) - (J_y - J_z)\,\dot{\psi}(\dot{\chi} + \varphi\dot{\psi}) - c_{12} y + c_{22}\varphi = -M\chi \tag{4.12a}$$

$$J_y(\ddot{\chi} + \varphi\ddot{\psi} + \dot{\varphi}\dot{\psi}) + (J_x - J_z)\,\dot{\psi}(\dot{\varphi} - \chi\dot{\psi}) + c_{12} x + c_{22}\chi = M\varphi \tag{4.12b}$$

$$J_z\left\{\ddot{\psi}+\frac{1}{2}(\chi\ddot{\varphi}-\varphi\ddot{\chi})-\dot{\psi}(\varphi\dot{\varphi}+\chi\dot{\chi})\right\}-$$

$$-(J_x-J_y)(\dot{\varphi}-\chi\dot{\psi})(\dot{\chi}+\varphi\dot{\psi})=M \qquad (4.12c)$$

The first two equations agree with the last two equations (3.20), whereas (4.12c) is the same as (3.19).

5. *The critical speeds.* We now return to the equations of motion (3.21) and first investigate whether a stationary movement of the system is possible, i. e. a movement for which $\dot{x}=\dot{y}=\dot{\varphi}=\dot{\chi}=0$. In this case the values of x, y, φ and χ are determined by the four homogeneous equations

$$(c_{11}-m\omega^2)x+c_{12}\chi=0, \quad c_{12}x+\{c_{22}-(J_x-J_z)\omega^2\}\chi=0 \qquad (5.1a)$$

$$(c_{11}-m\omega^2)y-c_{12}\varphi=0, \quad -c_{12}y+\{c_{22}-(J_y-J_z)\omega^2\}\varphi=0 \qquad (5.1b)$$

So we find that there are two kinds of critical speeds, one related to x and χ and the other to y and φ. We shall denote the critical speeds by ω_k and the displacements belonging to them by x_k, χ_k and y_k, φ_k respectively.

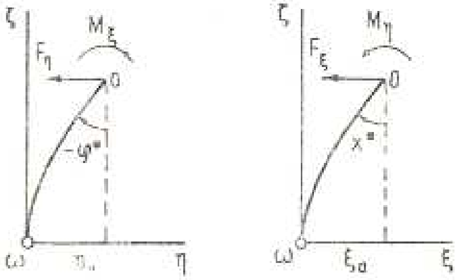


Fig. 3. The relations between the loads and the displacement quantities of the shaft.

Further on, it is advantageous to introduce reduced ("dimensionless") quantities. We first define the fundamental angular velocity ω_0 by

$$\omega_0=\sqrt{c_{11}/m} \qquad (5.2)$$

and we call

$$\alpha=c_{11}J_z/c_{22}m, \quad \gamma=c_{12}/\sqrt{c_{11}c_{22}}, \quad \underline{\omega}=\omega/\omega_0$$
$$\mu_x=J_x/J_z, \quad \mu_y=J_y/J_z \qquad (5.3)$$

For a cantilever shaft we have, according to (3.2): $\gamma^2=3/4$. Thus the critical speeds $\underline{\omega}_k$ are the roots of the equations

$$(1-\underline{\omega}^2)\{1-\alpha(\mu_x-1)\underline{\omega}^2\}-\gamma^2=0$$
$$(1-\underline{\omega}^2)\{1-\alpha(\mu_y-1)\underline{\omega}^2\}-\gamma^2=0 \qquad (5.4)$$

The equations can best be solved by first reducting them to the form

$$\mu_{x,y} = \frac{(1+\alpha\underline{\omega}^2)(1-\underline{\omega}^2)-\gamma^2}{\alpha\underline{\omega}^2(1-\underline{\omega}^2)} \qquad (5.5)$$

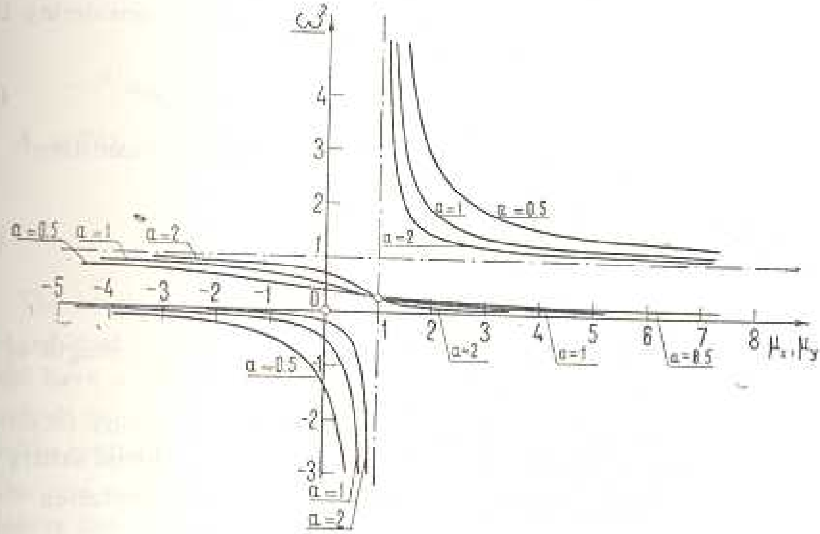


Fig. 4. The relations between $\mu_{x,y}$ and $\underline{\omega}^2$ for $\gamma^2=3/4$ and various values of α.

In fig. 4 we have plotted the relation between $\mu_{x,y}$ and $\underline{\omega}^2$ for various values of α. From this figure we conclude that a system has only one critical speed related to the motion (x, χ) when $0<\mu_x<1$, and two critical speeds related to this motion when $\mu_x>1$; the same holds for the motion (y, φ), which is determined by μ_y rather than by μ_x. We also see that the qualitative behaviour of the solution is not essentially influenced by the magnitude of the parameter α $(0<\alpha<\infty)$, neither by that of the parameter γ $(0<\gamma^2<1)$.

There are two cases which are especially important: the thin disk which is perpendicular to the axis oz, and the thin disk which contains this axis (so that we can choose the axis ox perpendicular to the plane of the disk). In the first case we have

$$J_z = J_x + J_y, \quad \mu_x + \mu_y = 1 \qquad (5.6a)$$

and in the second case

$$J_x = J_y + J_z, \quad \mu_x - \mu_y = 1 \qquad (5.6b)$$

From fig. 4 we now may conclude that in the first case never more than two critical speeds exist for the complete system, whereas in the second case there are three critical speeds for $\mu_y<1$ $(J_y<J_z)$ and four critical speeds for $\mu_y>1$ $(J_y>J_z)$. In the case of a two-blade propellor we have approximately $J_x = J_z > 0$, $J_y = 0$, $\mu_x = 1$, $\mu_y = 0$, so that there would be only two different critical speeds; in reality the propellor blades are not completely flat in the z-direction, so that $J_y > 0$, $J_x > J_z$, $\mu_x > 1$; thus three different critical speeds occur.

6. *The stability of the motion.* The stability of the solution of the equations of motion (3.21) can only be found by considering their solution. This can be represented by

$$x = x_0 e^{pt}, \quad y = y_0 e^{pt}, \quad \varphi' = \varphi_0 e^{pt}, \quad \chi = \chi_0 e^{pt} \tag{6.1}$$

Besides the parameters (5.3) we introduce the reduced quantities

$$\begin{gathered} \underline{p} = p/\omega_0, \quad \underline{x} = x/l, \quad \underline{y} = y/l \\ \underline{\varphi} = \frac{\varphi}{l}\sqrt{\frac{c_{22}}{c_{11}}}, \qquad \underline{\chi} = \frac{\chi}{l}\sqrt{\frac{c_{22}}{c_{11}}} \end{gathered} \tag{6.2}$$

and, in the same way, $\underline{x}_0$, $\underline{y}_0$, $\underline{\varphi}_0$, $\underline{\chi}_0$; l is a reference length of the order of x_0 and y_0.

Inserting the solution (6.1) into the equations of motion (3.21) and using the relations (6.2) we find that $\underline{x}_0$, $\underline{y}_0$, $\underline{\varphi}_0$, $\underline{\chi}_0$ should satisfy the four homogeneous linear equations which read in vector notation

$$\begin{pmatrix} \underline{p}^2+1-\underline{\omega}^2 & \gamma & -2\underline{\omega}\underline{p} & 0 \\ \gamma & 1+\alpha\mu_y\underline{p}^2-\alpha(\mu_x-1)\underline{\omega}^2 & 0 & -\alpha(1-\mu_x-\mu_y)\underline{\omega}\underline{p} \\ 2\underline{\omega}\underline{p} & 0 & \underline{p}^2+1-\underline{\omega}^2 & -\gamma \\ 0 & \alpha(1-\mu_x-\mu_y)\underline{\omega}\underline{p} & -\gamma & 1+\alpha\mu_x\underline{p}^2-\alpha(\mu_y-1)\underline{\omega}^2 \end{pmatrix} \begin{pmatrix} \underline{x}_0 \\ \underline{\chi}_0 \\ \underline{y}_0 \\ \underline{\varphi}_0 \end{pmatrix} = \overline{0} \tag{6.3}$$

The values of $(\underline{x}_0, \underline{\chi}_0, \underline{y}_0, \underline{\varphi}_0)$ can only be unequal to zero if $\underline{p}$ is a root of the characteristic equation of the system (6.3). It is easily found that this is a fourth degree equation in the unknown $\underline{p}^2$.

This can be shown by introducing the auxiliary quantities

$$\begin{gathered} a = \alpha\mu_x, \quad b = \alpha\mu_y, \quad c = \alpha(1-\mu_x-\mu_y) \\ m = 1+\alpha(1-\mu)\underline{\omega}^2, \quad n = 1+\alpha(1-\mu)\underline{\omega}^2 \end{gathered} \tag{6.4}$$

Thus the characteristic equation reads

$$\begin{vmatrix} \underline{p}^2+1-\underline{\omega}^2 & \gamma & -2\underline{\omega}\underline{p} & 0 \\ \gamma & b\underline{p}^2+m & 0 & -c\underline{\omega}\underline{p} \\ 2\underline{\omega}\underline{p} & 0 & \underline{p}^2+1-\underline{\omega}^2 & -\gamma \\ 0 & c\underline{\omega}\underline{p} & -\gamma & a\underline{p}^2+n \end{vmatrix} = 0 \tag{6.5}$$

or after evaluation:

$$\sum_{q=0}^{4} A_q \underline{p}^{2q} = 0 \tag{6.6}$$

with

$$A_0 = \{(1-\underline{\omega}^2)\, m - \gamma^2\}\{(1-\underline{\omega}^2)\, n - \gamma^2\}$$

$$A_1 = (1-\underline{\omega}^2)^2 (am + bn + c^2\underline{\omega}^2) -$$

$$-\gamma^2\{a + b + m + n + (4c - a - b)\,\underline{\omega}^2\} + 2mn(1+\underline{\omega}^2)$$

$$A_2 = 2(1+\underline{\omega}^2)(am + bn + c\underline{\omega}^2) + ab(1-\underline{\omega}^2)^2 - \gamma^2(a+b) + mn$$

$$A_3 = am + bn + c^2\underline{\omega}^2 + 2ab(1+\underline{\omega}^2)$$

$$A_4 = ab$$

The solution (6.1) is only stable if all the eight roots $\underline{p}_k$ are in the left-hand part of the complex plane or on the imaginary axis: no root should have a positive real part. This means that the quantities $\underline{p}_k^2$ should all be real and negative. A boundary between a stable and an unstable region is formed either when $\underline{p}_k^2$ goes from the negative part of the real axis through the origin to the positive part of this axis, or leaves the negative part of the real axis and becomes complex; as always two roots $\underline{p}_k^2$ of (6.6) are complex adjungate, in the latter case on the stability boundary two roots $\underline{p}_k^2$ then coincide.

The first kind of stability boundary is found very easily. In that case we have to substitute $\underline{p}=0$ into (6.5). But then we have the same situation as in section 5, and the characteristic equation reduces to the set of the two equations (5.4), which is equivalent to the equation $A_0=0$ (see 6.7). So the curves which indicate the critical speed are at the same time boundaries between a stable and an unstable region.

We have investigated the cases $\mu_x + \mu_y = 1$ (thin disk perpendicular to the shaft) and $\mu_x - \mu_y = 1$ (thin disk in the plane of the shaft) more in detail by calculating the roots for certain values of μ (in each case for $\alpha = 2$ and $\gamma^2 = 3/4$).

In the first case all the four roots $\underline{p}_k^2$ are always real and we find only two stability boundaries, coinciding with the curves of critical speed: see fig. 6. Two of the roots are shown in fig. 5 for the value $\mu = \mu_y/\mu_x = 2/3$, the two other being more to the left in the coordinate plane. We find that instability occurs when the most-right root $\underline{p}_k^2$ crosses the $\underline{\omega}$ axis and becomes positive.

In the second case there are three regions of instability (fig. 8). The regions Ia and Ib are limited by the curves on which the speed is critical, as is shown in fig. 7 (for $\mu = 2/3$), where the roots $\underline{p}_1^2$ and $\underline{p}_2^2$ cross the $\underline{\omega}$ axis in points corresponding with these curves. But there is another region of instability (II), where two roots p_λ^2 become complex, which also is shown in fig. 7. The latter region can only be found by calculating the roots p_λ^2 and determining the values of ω where two roots p_λ^2 coincide.

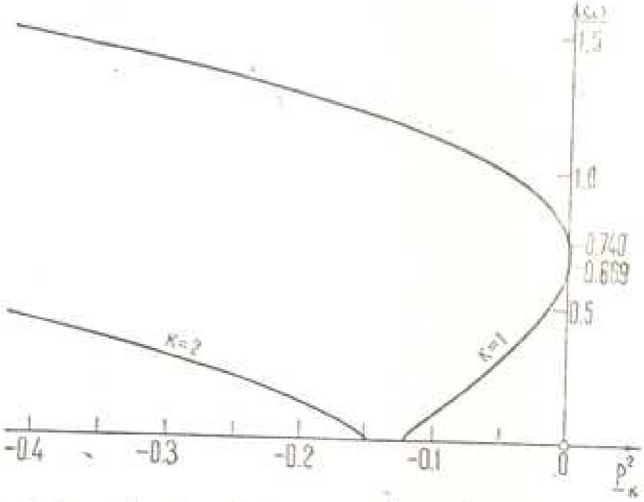


Fig. 5. The behaviour of the roots p_1^2 and p_2^2 as function of ω for $\mu_x+\mu_y=1$, $\mu=2/3$ and $\gamma^2=3/4$.

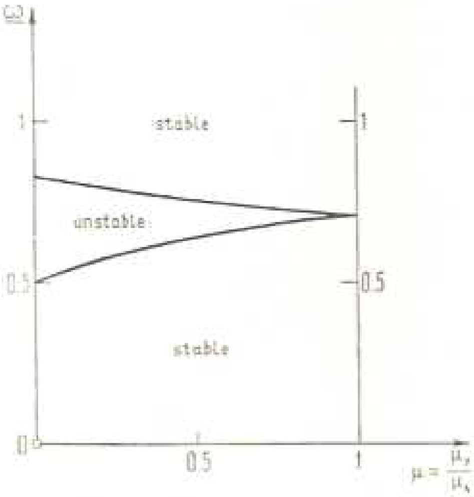


Fig. 6. The stability regions for $\mu_x+\mu_y=1$, $\varkappa=2$ and $\gamma^2=3/4$.

That the curves of critical speeds partially constitute the stability boundaries, can also be shown by considering the influence of the gyroscopic forces: this enables us to check one of the results of the investigations of Rayleigh and Četaev [8]. We shall do so in the next section.

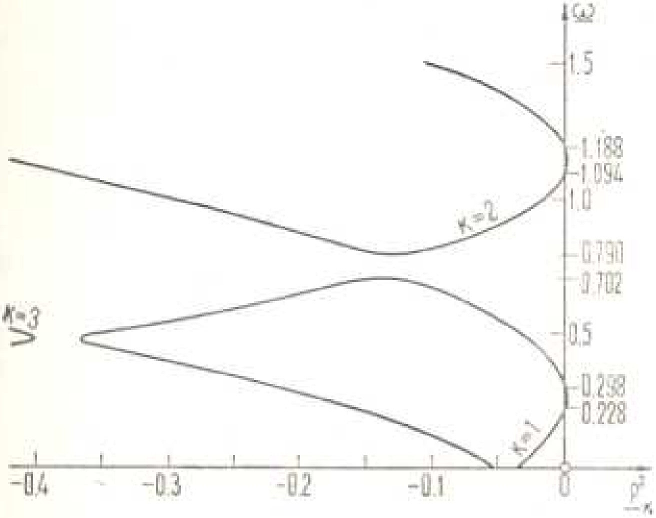


Fig. 7. The behaviour of the roots p_k^2 as function of μ for $\mu_x=\mu_y=1$, $\alpha=2/3$, $s=2$ and $\gamma^2=3/4$.

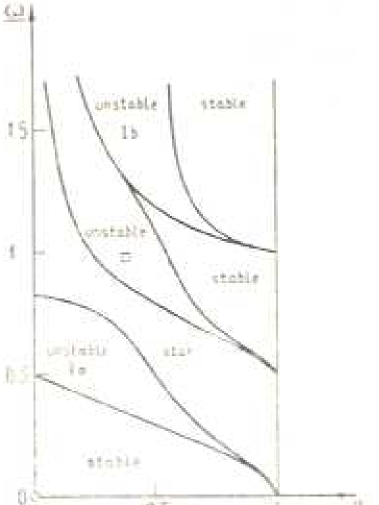


Fig. 8. The stability regions for $\mu_x=\mu_y=1$, $s=2$ and $\gamma^2=3/4$.

7. *The relation between the curves of critical speed and the stability boundaries.* It is possible to consider the equations of motion **(3.21)** as Lagrange's equations

$$\frac{d}{dt}\frac{\partial T^*}{\partial \dot{q}_i}-\frac{\partial T^*}{\partial q_i}+\frac{\partial U^*}{\partial q_i}=Q_i^* \tag{7.1}$$

derived from the modified kinetic energy

$$T^* = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) + \frac{1}{2} J_x \dot{\varphi}^2 + \frac{1}{2} J_y \dot{\chi}^2 \tag{7.2a}$$

the modified potential energy

$$U^* = \frac{1}{2} (c_{11} - m\omega^2)(x^2 + y^2) + c_{12}(x\chi - y\varphi) +$$
$$+ \frac{1}{2} \{c_{22} - (J_y - J_z)\omega^2\} \varphi^2 + \frac{1}{2} \{c_{22} - (J_x - J_z)\omega^2\} \chi^2 \tag{7.2b}$$

and the modified generalised forces

$$Q_x^* = 2m\omega\dot{y}, \quad Q_y^* = -2m\omega\dot{x}$$
$$Q_\varphi^* = (J_x + J_y - J_z)\omega\dot{\chi}, \quad Q_\chi^* = -(J_x + J_y - J_z)\omega\dot{\varphi} \tag{7.2c}$$

Here the kinetic energy T^* is a homogeneous quadratic positive definite function of the velocities $\dot{x}$, $\dot{y}$, $\dot{\varphi}$ and $\dot{\chi}$ and the potential energy U^* is a homogeneous quadratic, but not always positive definite, function of the coordinate x, y, φ and χ; the generalised forces Q_i^* have the character of gyroscopic forces. Thus, the power of the generalised forces

$$P^* = Q_x^*\dot{x} + Q_y^*\dot{y} + Q_\varphi^*\dot{\varphi} + Q_\chi^*\dot{\chi} \tag{7.3}$$

is equal to zero. Because T^* is a homogeneous quadratic function, now the energy balance

$$\frac{d}{dt}(T^* + U^*) = P^* \tag{7.4}$$

(with $P^* = 0$) holds, and we see that the stability of the motion depends on the nature of U^*: when U^* is positive definite, the motion is certainly stable, whereas its nature is uncertain when U^* is not positive definite. Četaev [8] has shown that for such an (undamped) system the gyroscopic forces can stabilise the system in a certain domain of the region where the potential energy is not positive definite.

To find the boundaries separating the regions where U^* is positive definite from the regions where this is not the case, we have to calculate the eigenvalues of the matrix $\bar{C}^*$ of the potential energy, the expression of which can be written as

$$U^* = \frac{1}{2} \tilde{\bar{q}} \bar{C}^* \bar{q} \tag{7.5}$$

From (7.2b) we find

$$\overline{C}^* = \begin{vmatrix} c_{11} - m\omega^2 & c_{12} & 0 & 0 \\ c_{12} & c_{22} - (J_x - J_z)\omega^2 & 0 & 0 \\ 0 & 0 & c_{11} - m\omega^2 & -c_{12} \\ 0 & 0 & -c_{12} & c_{22} - (J_y - J_z)\omega^2 \end{vmatrix} \tag{7.6}$$

By comparing this expression with the matrix of the combined equations (5.1a) and (5.1b) we easily find that the values of ω where an eigenvalue of $\overline{C}^*$ is equal to zero, are the roots of the equations (5.4) for the critical speed of the system, in agreement with our previous result. For the further discussion it is advantageous to use again reduced quantities; by means of (5.3) and (6.2) we find for the characteristic equation for C^* in such quantities

$$\begin{vmatrix} 1 - \underline{\omega}^2 - \underline{\lambda} & \gamma & 0 & 0 \\ \gamma & 1 - \alpha(\mu_x - 1)\underline{\omega}^2 - \underline{\lambda} & 0 & 0 \\ 0 & 0 & 1 - \underline{\omega}^2 - \underline{\lambda} & -\gamma \\ 0 & 0 & -\gamma & 1 - \alpha(\mu_y - 1)\underline{\omega}^2 - \underline{\lambda} \end{vmatrix} = 0 \tag{7.7}$$

Thus the values of $\underline{\lambda}$ are determined by

$$\underline{\lambda}^2 - c_1\underline{\lambda} + c_0 = 0 \tag{7.8}$$

with

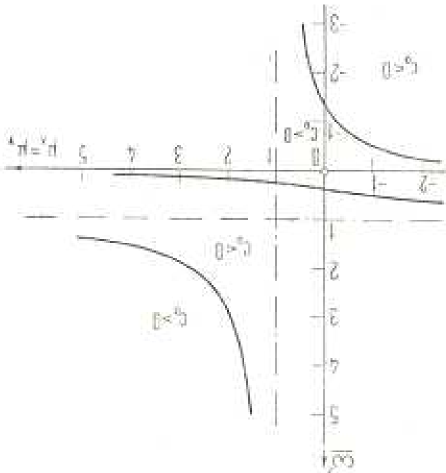
$$\begin{aligned} c_0 &= (1 - \underline{\omega}^2)\{1 - \alpha(\mu_{x,y} - 1)\underline{\omega}^2\} - \gamma^2 \\ c_1 &= 1 - \underline{\omega}^2 + 1 - \alpha(\mu_{x,y} - 1)\underline{\omega}^2 \end{aligned} \tag{7.9}$$

For stability it is necessary that both $c_0 > 0$ and $c_1 > 0$.

For c_0 this is indicated in fig. 9a—9b, which correspond with fig. 4 for the special values of $\alpha = \frac{1}{2}$ and $\gamma = 2$. For c_1 we have drawn up an analogous set of figures: fig. 10 and fig. 11a—11b. Comparing the four figures we find that in the only quadrant which has a physical significance ($\mu_{x,y} > 0$, $\underline{\omega}^2 > 0$) there is always a stable region situated against the $\mu_{x,y}$—axis, whereas for certain values of α a second stable region is possible (such as in the case of fig. 8).

In this way it is possible to determine the regions where the motion is certainly stable. In the regions where one or more values of λ are negative, the motion can either be stable or unstable. This is demonstrated by comparing the figures 4, 10, 11a and 11b. Here we find that in the unstable regions Ia and Ib one eigenvalue is negative, whereas in the domain consisting of the unstable region II and the two adjacent stable regions two eigenvalues $\underline{\lambda}$ are so. In this latter domain, with the exception of region II the unstability is removed by the effect of the gyroscopic forces.

The figures 4, 10, 11a and 11b enable us for the case of a cantilever shaft to determine the region where the motion is stable



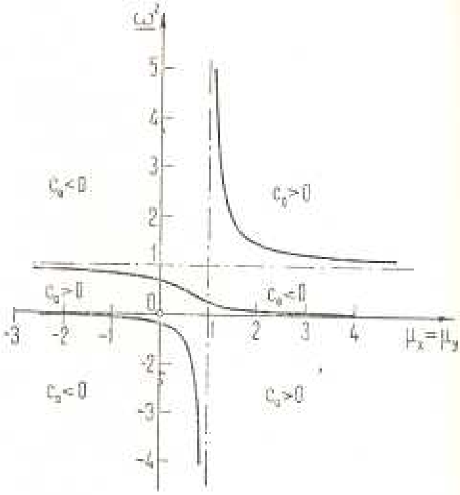


Fig. 9a. The sign of c_0 for $\alpha=1/2$ and $\gamma^2=3/4$. Fig. 9b. The sign of c_0 for $\alpha=2$ and $\gamma^2=3/4$.

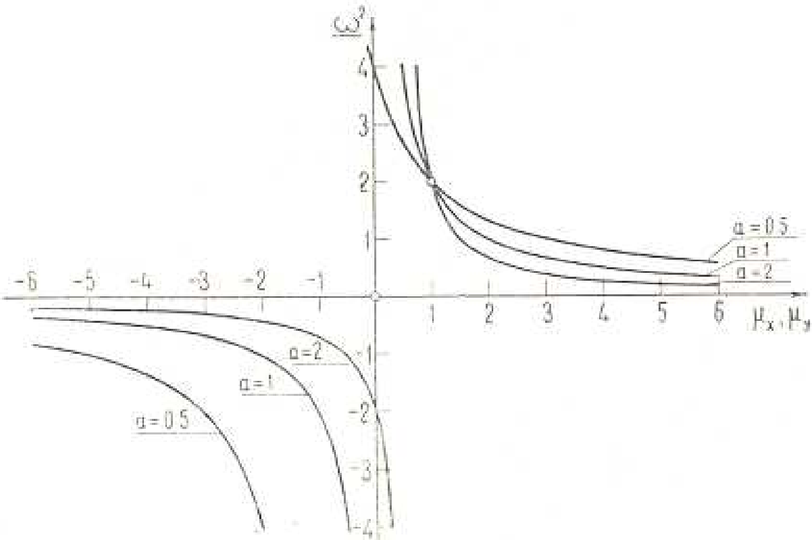


Fig. 10. The curves $c_1=0$ for $\gamma^2=3/4$ and various values of α.

anyhow. For finding the nature of the motion in the remaining regions it is necessary to calculate the roots of the characteristic equation (6.6).

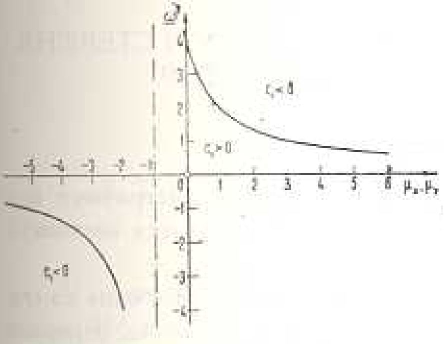


Fig. 11a. The sign of c_1 for $\alpha=1/2$ and $\gamma^2=3/4$.

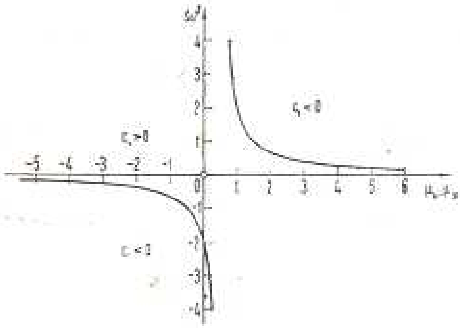


Fig. 11b. The sign of c_1 for $\alpha=2$ and $\gamma^2=3/4$.

8. *Conclusions.* By means of the results of the previous sections it is possible to find the stability of a cantilever shaft with a symmetrical cross-section, bearing an unsymmetrical body. It is not difficult to extend the investigation to the case of a shaft which is supported in other ways, because it is only necessary to change the value of the parameter γ.

It would be quite interesting to extend the investigation to the case of (external and internal) damping.

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ՈՉ ՍԻՄԵՏՐԻԿ ՌՈՏՈՐ ԿՐՈՂ ՊՏՏՎՈՂ ՁՈՂԻ ՇԱՐԺՄԱՆ ԿԱՅՈՒՆՈՒԹՅԱՆ ՎԵՐԱԲԵՐՅԱԼ

Ամփոփում

Ուսումնասիրվում է անզանգված առաձգական պտտվող և սիմետրիկ լայնական կտրվածք ունեցող ձողի շարժումը, երբ նա կրում է ոչ սիմետրիկ մարմին։

Մարմնի զանգվածի կենտրոնը դասավորված է ձողի պտտման առանցքի վրա և իներցիայի գլխավոր առանցքներից մեկը համընկնում է այդ առանցքի հետ ձողի չդեֆորմացված վիճակում։ Շարժման հանդարտումը հաշվի չի առնվում։ Ձողի պտտման անկյունային արագությունը հաստատուն է։

Գտնվել են երկու տեսակ անկայուն շրջանակներ՝ մեկը սահմանափակված է կրիտիկական արագությունների միատեսին համապատասխանող կորերով, մյուսը կորերով, որոնց վրա որոշիչ հավասարման արմատների երկու զույգից համընկնում են երկուսը, իսկ երկուսն էլ մաքուր կեղծ են։ Արդյունքները համաձայնեցվում են Ռելեյի և Ջենտայի ձևակերպումների հետ։

А. Д. де ПАТЕР, А. ХУГЕРВАРД

ОБ УСТОЙЧИВОСТИ ДВИЖЕНИЯ ВРАЩАЮЩЕГОСЯ СТЕРЖНЯ, НОСЯЩЕГО НЕСИММЕТРИЧНЫЙ РОТОР

Резюме

Исследуется движение вращающегося, упругого, с симметричным поперечным сечением стержня без массы, когда на нем находится несимметричное тело.

Центр массы тела расположен на оси вращения стержня и одна из его главных осей инерции совпадает с этой осью стержня в его недеформированном состоянии. Затухание движения не принимается во внимание. Угловая скорость вращения стержня является постоянной.

Найдено два вида областей неустойчивости: одна ограничена кривыми, которые соответствуют системе критических скоростей, другая — кривыми, на которых из двух пар корней характеристического уравнения два корня совпадают, а два других являются чисто мнимыми.

Результаты согласуются с формулировками Релея и Четаева.

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